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A FREQUENTIST SELECTION-OF-THE-BEST PROCEDURE WITHOUT INDIFFERENCE ZONE

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ABSTRACT

Many procedures have been proposed in the literature to select the best from a finite set of alternatives. Among these procedures, frequentist procedures are typically designed under an indifference-zone (IZ) formulation, where an IZ parameter needs to be specified by users at the beginning of procedures. The IZ parameter is the smallest difference in the performance measure that the users care. In practice, however, the IZ parameter is often difficult to specify appropriately and may be specified in a conservative way (thus leading to excessive sampling effort). In this paper, we propose a frequentist IZ-free selection-of-the-best procedure. The procedure guarantees to select the best with at least a pre-specified probability of correct selection in an asymptotic regime. Through numerical studies, we show our procedure may out-perform IZ procedures, in terms of total sample size, when the IZ parameter is set conservatively or there are a large number of alternatives.

1 INTRODUCTION

Many selection procedures have been proposed in the literature to select the best from a finite set of alternative systems, where the best is defined as the system with the largest (or smallest) mean performance (see Bechhofer and Goldsman (1995) for a summary). Among these procedures, there are frequentist and Bayesian procedures. Frequentist procedures, such as those proposed by Rinott (1978) and Kim and Nelson (2001), allocate samples to different systems in order to achieve a guaranteed lower bound for the probability of correct selection (PCS) even for the least favorable configuration. They are typically conservative (i.e., requiring more samples than necessary) for an average case. Bayesian procedures, such as those proposed by Chen et al. (2000) and Chick and Inoue (2001), allocate a finite computing budget to different systems in order to either maximize the posterior (Bayesian) PCS or minimize the opportunity cost. They often require fewer samples than frequentist procedures, but typically do not provide a guaranteed (frequentist) PCS. In this paper we take the frequentist viewpoint to design selection-of-the-best procedures with a guaranteed PCS.

There are two frequentist's formulations in the literature to solve the selection-of-the-best problem. One is the subset-selection formulation of Gupta (1956) and Gupta (1965). It seeks to select a random subset of the alternative systems that contains the best, regardless of the configurations of the means of these systems. Besides the procedures proposed by Gupta (1956) and Gupta (1965), the procedures of Hsu and Panchapakesesan (1973) and Sullivan and Wilson (1989) are also designed under this formulation. The major drawback of the subset-selection formulation is that those procedures may select a subset which has more than one alternative system in it. In such cases the procedures cannot tell which system is the best. This may create problems for users because their goal is often to find the best system instead of a subset that contains the best. The other formulation is the indifference-zone (IZ) formulation of Bechhofer

(1954). It seeks to select the single best system whenever the difference of means between the best and the second-best systems is at least δ, where $\delta > 0$ is termed as the IZ parameter and often refers to the smallest difference worth detecting. The IZ formulation appears more popular than the subset-selection formulation, because it returns only a single best system. There are many IZ procedures, such as Rinott's procedure (Rinott 1978), Paulson's procedure (Paulson 1964) and KN procedure (Kim and Nelson 2001), and some of these procedures have been implemented by commercial simulation software packages, such as Arena.

However, when using IZ procedures in practice, it is often difficult for users to specify an appropriate IZ parameter δ. Let Δ denote the difference of the means between the best and the second-best systems. Under the IZ formulation of Bechhofer (1954), procedures provide a guaranteed PCS in the condition that the difference Δ is at least δ. Hence, they require users to specify an appropriate δ (which is smaller than Δ) at the beginning of procedures. Unfortunately, it is not a trivial task in practice due to the lack of prior knowledge about the difference Δ. Besides, in some cases, systems are revealed sequentially (see, for instance, Hong and Nelson 2007), resulting in a gradually changing Δ. As a remedy, one may choose a conservative δ, which is close to zero and thus may be smaller than the unknown Δ. Notice that most IZ procedures are designed as if δ is exactly the true difference Δ. Hence, the users have to suffer from excessive sampling efforts by choosing δ in a conservative way.

Even if the IZ parameter δ is chosen appropriately, users may still encounter a risk of inefficiency. IZ procedures often break down a selection-of-the-best problem into a group of pairwise comparisons and treat δ as the difference worth detecting in all pairwise comparisons. However, δ is chosen only based on Δ, which is essentially the difference to detect in the pairwise comparison between the best and the second-best systems. It implies that, an appropriate δ cannot avoid the users from excessive sampling in those pairwise comparisons between the best and the other systems except for the second-best one. The conservativeness tends to be serious in problems where the number of alternative systems is large, see e.g., Luo and Hong (2011) and Ni, Hunter, and Henderson (2013). In such cases, there are often many inferior systems whose means are significantly worse than that of the best system. This makes IZ procedures inefficient.

In this paper we target at designing frequentist selection-of-the-best procedures that are IZ-free. When the IZ parameter is absent, the procedures do not have the smallest difference worth detecting. Therefore, the procedures are intended to select the best system. Consider a problem of only two systems and the difference of their means is Δ. Then, the target procedures seek to eliminate the inferior system as long as $\Delta \neq 0$, no matter how close Δ is to 0. It appears impossible for two-stage procedures to achieve this target because it may need infinite samples. To see this, the sample sizes required by two-stage procedures are negatively proportional to Δ. Because Δ may be arbitrarily close to 0, the two-stage procedures need an infinite sample size without a prior knowledge on the size of Δ. To solve this problem, we resort to sequential procedures in this paper. Sequential procedures often approximate the difference between two systems as a Brownian motion process with an unknown drift. In our two-system problem, the drift is Δ. Then, to select out the better system, it is equivalent to consider whether the drift Δ is zero or not. In other words, we need to construct a continuous region which can distinguish a Brownian motion with no drift from that with any unknown nonzero drift. The principal difference of these two Brownian motions is the rates at which they grow to infinity. Particularly, the former approaches to infinity at the rate bounded by $O(\sqrt{t\log\log t})$ due to the law of iterative logarithm (Durrett 2010), while the latter approaches to infinity at the rate bounded by $O(t)$. Driven by this insight, we propose a continuation region formed by the boundaries which approach to infinity at a rate $O(\sqrt{t\log t})$, which is between $O(\sqrt{t\log\log t})$ and $O(t)$. With the region formed by such boundaries, our sequential procedure eliminates the inferior system (or does nothing) if the approximated Brownian motion exits from (or stays all the time in) the region.

For a general continuation region, it is hard, even impossible, to explicitly and exactly analyze the first exit time of a Brownian motion (see Durbin 1985). It appears hopeless to prove the statistical validity of our procedure through a finite-sample analysis. Thus, we consider an asymptotic regime where the prescribed

PCS goes to 1. This asymptotic regime dates back to Perng (1969) and Dudewics (1969). In this regime, we are able to prove that our procedure is asymptotically valid, in the sense that the PCS achieved by our procedure grows to 1 as fast as the prescribed PCS does. The asymptotic statistical validity provides a theoretical support as well as a practical guidance to use our procedure.

In summary, we design a sequential frequentist IZ-free selection-of-the-best procedure. This procedure uses a first stage to estimate the variances of the systems and determine a continuation region as discussed above for each pair of systems; it then takes samples sequentially to update the variance estimators and eliminate inferior systems based on the continuation regions. We prove its statistical validity in the asymptotic regime when PCS goes to 1. In the meanwhile, we propose a truncated sequential IZ-free procedure for the case when the best systems are tied or the computing budget is limited. We show that the selected system by the truncated procedure lies within a certain "distance" of the best system with a probability guarantee.

Furthermore, we compare our procedure theoretically and experimentally with a leading IZ procedure, i.e., the *KN*++ procedure of Kim and Nelson (2006), in terms of sample sizes required to select the best system with the same PCS. We provide an analytic expression for the sample size needed by our procedure, and find that our procedure out-performs the *KN*++ procedure when the IZ parameter is set to be smaller than one fifth of the true difference. Besides, we compare our procedure with the *KN*++ procedure through numerical studies and the results are consistent with the theory. In addition, when there are a large number (e.g. 500) of alternative systems with monotone configurations of means, our procedure appears significantly more efficient than the *KN*++ procedure.

The rest of our paper is organized as follows. In Section 2, we propose a sequential IZ-free procedure when the samples from all systems are assumed to be normal distributed with known variances. In Section 3, we relax the assumption and design a sequential procedure for a more general setting. The asymptotic statistical validities of both procedures are provided, respectively, in Sections 2 and 3. In Section 4, we analyze the asymptotic average sample size required by our procedures and compare it with the *KN*++ procedure. In Section 5, numerical experiments are conducted to show the performance of our procedures.

2 SEQUENTIAL PROCEDURE FOR NORMAL OBSERVATIONS WITH KNOWN VARIANCES

Suppose there are k alternative systems in contention at the beginning of selection, and the best system has the largest mean performance. Hereafter we postulate that the IZ parameter is hard to specify, since there is often no prior knowledge to users about the difference between the largest and the second-largest mean performances. In other words, the IZ parameter is not available when we design procedures.

To facilitate our presentation, we denote X_{ij} as the jth independent observation from System i, for $i = 1,2,\ldots,k$ and $j = 1,2,\ldots$. Denote μ_i and σ_i^2 as the mean and variance of the simulated System i. Moreover, assume that $\mu_1 > \mu_2 \geq \ldots \geq \mu_k$. Therefore, our goal is to select System 1. It is worth noting that there is no requirement on the difference between μ_1 and μ_2. To focus on the key idea of designing an IZ-free procedure, we assume, in this section, that X_{ij} follows a normal distribution with a known variance σ_i^2, for all $i = 1,2,\ldots,k$.

2.1 Key Idea

Our goal is to design a sequential IZ-free procedure which guarantees to select System 1 with a probability at least $1-\alpha$ $(0 < \alpha < 1-1/k)$. Notice that a sequential procedure often approximates the partial sum process of the difference of the means between two systems by a Brownian motion process and design a continuation region to determine the first exit time of the Brownian motion. Hence, to design a sequential procedure, we need to address at least two issues: (1) justifying a Brownian approximation to the partial sum of difference; (2) finding a proper continuation region. Under the assumptions of normality and known variances, the partial sum process of the differences of the means between any two systems can be viewed as a Brownian motion process observed at a set of discrete time points $\{1,2,\ldots\}$. Therefore, the first issue is tackled.

The second issue is tackled as follows. As discussed in Selection 1, without an IZ parameter, the designed continuation region is intended to distinguish a Brownian motion process with no drift from a Brownian motion process with any unknown nonzero drift. The principal difference of them is that the former approaches to infinity at the rate of $O(\sqrt{t\log\log t})$ due to the law of iterative logarithm (Durrett 2010), while the latter approaches to infinity at the rate of $O(t)$. We propose a continuation region as $[-g(t),g(t)]$, where $g(t)=\sqrt{[a+\log(t+1)](t+1)}$, which goes to infinity at a rate between $O(\sqrt{t\log\log t})$ and $O(t)$.

Jennen and Lerche (1981) provide an asymptotic analysis for the first exit time of a Brownian motion from the continuation region formed by $\pm g(t)$. Their result (stated as Lemma 1) enables us to establish the statistical validity of a procedure that uses the continuation region.

Lemma 1 (Jennen and Lerche 1981) Let $g(t)=\sqrt{[a+\log(t+1)](t+1)}$ be a non-negative continuous function and $B(t)$ be a Brownian motion with no drift. Then, $\mathbb{P}(B(T)<-g(T))\doteq\frac{1}{2}e^{-a/2}$, where $T=\inf\{t:|B(t)|\geq g(t)\}$. Hereafter we use the symbol "$\doteq$" to denote the asymptotic equivalence with respect to (w.r.t.) a. In particular, $f(a)$ is asymptotic equivalent to $g(a)$ w.r.t. a, i.e., $f(a)\doteq h(a)$, if $\lim_{a\to\infty} f(a)/g(a)=1$.

2.2 Procedure

In this subsection, we propose a sequential procedure that uses the continuation region denoted by $\pm g(t)$. The procedure, like the *KN* procedure, eliminates inferior systems sequentially, and terminates when only one system is left.

Procedure 1

Setup. Select the overall desired PCS $1-\alpha(0<\alpha<1-1/k)$ and let $a=-2\log\left(2\alpha/(k-1)\right)$.

Initialization. Let $I=\{1,2,\ldots,k\}$ be the set of systems still in contention. Set $n=1$.

Screening. Obtain one additional output X_{in} from System i, for all $i\in I$. Define $I^{\text{old}}=I$ and $I=I^{\text{old}}\setminus\left\{i\in I^{\text{old}}:Z_{ij}(n)<-g_{ij}(n)\text{ for some other } j\in I^{\text{old}}\right\}$, where $Z_{ij}(n)=\sum_{r=1}^{n}(X_{ir}-X_{jr})$ is the sum of the difference between Systems i and j, and $g_{ij}(n)=\sqrt{(\sigma_i^2+\sigma_j^2)[a+\log(n+1)](n+1)}$ is the boundary that we use to eliminate inferior systems.

Stopping Rule. If $|I|=1$, then stop and select the system whose index is in I as the best. Otherwise, set $n=n+1$ and go to **Screening**.

Consider a pairwise comparison between Systems i and j. Assume that System i has a larger mean than System j has and thus our procedure needs to eliminate System j. To achieve this target, we construct a continuation region in Procedure 1 to determine the stopping time of the partial sum process of their differences: terminate and eliminate the System i (System j) if this process exits the region from below (above), see Figure 1. An incorrect selection event occurs if System i is eliminated.

2.3 Statistical Validity

In this subsection we establish the statistical validity of Procedure 1 in the asymptotic regime where the PCS goes to 1 (i.e., $\alpha\to 0$). To achieve this goal, we need the following lemmas. Due to the page limit, we omit the proofs of these lemmas in this paper.

When implementing Procedure 1, an incorrect selection event is defined as an event that a Brownian motion with a positive drift exits the continuation region from below. Lemma 1 gives the probability of a Brownian motion with no drift exiting from the continuation region, and the following lemma shows that this probability is an upper bound for the probability of an incorrect selection event.

Lemma 2 Let $B(\cdot)$ and $B_\Delta(\cdot)$ denote a Brownian motion with no drift and a positive drift Δ, respectively, and let T and $\tilde{T}$ be the stopping times that $B(\cdot)$ and $B_\Delta(\cdot)$ exit the region $[-g(t),g(t)]$, where $g(t)$ is a continuous and non-negative function. Then, we have that $\mathbb{P}(B_\Delta(\tilde{T})<-g(\tilde{T}))\leq\mathbb{P}(B(T)<-g(T))$.

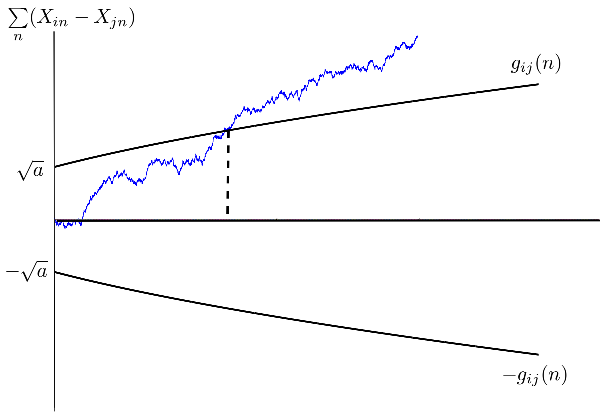

Figure 1: continuation region for Sequential IZ-Free Procedure

However, we only collect observations in the discrete time points in the procedure. Hence, a Brownian motion process is only an approximation to the discrete process obtained by taking values in a set of discrete times. The following lemma states that, under very general conditions, the procedure designed for the continuous Brownian motion process provides an upper bound on the probability of incorrect selection for the discrete process.

Lemma 3 (Jennison, Johnston, and Turnbull 1980) Suppose that a continuation region R is formed by the non-negative function $g(t)$ and $-g(t)$, $t \geq 0$. Let $B_\Delta(t)$ denote a Brownian motion with drift Δ, $\Delta > 0$, and $T_C = \inf\{t > 0 : B_\Delta \notin R\}$. A discrete process is obtained by observing B_Δ at a random, increasing sequence of times $\{t_i : i = 1, 2, \ldots\}$ takes values in a given countable set. The value of t_i depends on $B_\Delta(t)$ only through its values in the period $[0, t_{i-1}]$. We define $T_D = \inf\{t_i : B_\Delta \notin R\}$ and assume that $T_D < \infty$ a.s. If the conditional distribution of $\{t_i\}$ given $B_\Delta(t_i) = b$ is the same as that given $B_\Delta(t_i) = -b$, then $\mathbb{P}(B_\Delta(T_D) \leq -g(T_D)) \leq \mathbb{P}(B_\Delta(T_C) \leq -g(T_C))$.

With these lemmas above, we establish the asymptotic statistical validity of Procedure 1 and summarize it in the following theorem.

Theorem 1 Suppose that $X_{ij} (i = 1, 2, \ldots, k, j = 1, 2, \ldots)$ are independent normal random variables with unknown means μ_i and known variances σ_i^2. Without loss of generality, assume that $\mu_1 > \mu_2 \geq \ldots \geq \mu_k$. Then Procedure 1 selects System 1 with, $\limsup_{\alpha \to 0} \mathbb{P}(\text{ICS})/\alpha \leq 1$, where 'ICS' is defined as the event that System 1 is not selected as the best.

2.4 Truncated Procedure with Termination Point

Procedure 1 can be viewed as an "open" sequential procedure, in the sense that the number of samples taken before it terminates is not bounded. In practice, it is naturally expected that an "open" procedure takes a finite number of samples at the termination almost surely. Now we use the following theorem to show our procedure has this feature in each pairwise comparison, in the case that a difference does exist between the means of these two competing systems. In this theorem, we analyze a Brownian motion process instead of the partial sum process, because the Brownian motion process is more convenient under the normality assumption.

Theorem 2 Let $B_\Delta(t)$ denote a Brownian motion with a nonzero drift Δ $(\Delta \neq 0)$. Define $T_\Delta = \inf\{t : |B_\Delta(t)| \geq g(t)\}$. If $g(t) = o(t)$ (i.e., $\lim_{t\to\infty} g(t)/t = 0$), then $\mathbb{P}(T_\Delta < \infty) = 1$.

Notice that in Procedure 1 the boundary $g(t) (= \sqrt{[a + \log(t+1)](t+1)}) = o(t)$. Then, Theorem 2 implies that our procedure terminates with a finite number of samples almost surelywhen the two competing systems have different means. However, it is possible in practice that the alternative systems may have the tied "best" mean performance. In this case, our procedure may not terminate with a finite number of samples. To solve this problem, we propose a truncated sequential procedure, which appends a termination point N to Procedure 1. Particularly, in the truncated procedure, users terminate the procedure at time N if there are more than one remaining system; then select the remaining system with the largest sample mean as the best. Then, the sample size required to select the best system by the truncated procedure is bounded by N.

As a consequence, an early termination of the truncated procedure at N may result in a selected system that is not the best. Considering this, we address, in the rest of this subsection, the issue of telling users the "distance" between the true best system and the system selected by the procedure. It is conceivable that the distance depends on the termination point N. Roughly speaking, the later users choose to terminate the procedure, the more information the truncated procedure can obtain about the alternative systems and thus the nearer the selected system is to the true best system. Moreover, the following theorem provides a rigorous quantification of the distance when the procedure stops at N.

Theorem 3 Suppose that $X_i, i=1,2,\ldots,k$ are independent normal random variables with unknown means μ_i and known variances σ_i^2. Without loss of generality, assume that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$. Let N be the termination point that users choose and let $[1]$ denote the system selected by the truncated procedure. Denote $1-\alpha$ as the prescribed PCS and $d = N^{-1}\sqrt{\max_{j\neq i}(\sigma_i^2+\sigma_j^2)[a+\log(N+1)](N+1)}$. Then, we have, $\liminf_{\alpha\to 0}\mathbb{P}(\mu_1-\mu_{[1]}\leq d)/(1-\alpha)\geq 1$.

It is straightforward to find that d decreases as the termination point N increases. In other words, the later the procedure terminates, the better the selected system is.

3 SEQUENTIAL PROCEDURE UNDER GENERAL ASSUMPTIONS

In the previous section, we propose a sequential procedure (with or without a termination point) to select the best system without an IZ parameter and justify its statistical validity in the asymptotic regime, under the assumptions that the samples from all systems are independent and normally distributed with known variances. In this section, we relax these assumptions and provide an asymptotically valid procedure.

Let X_{ij} denote jth independent observation from System i. We assume that X_{ij} follows a general distribution with unknown mean μ_i and finite variance σ_i^2, for all i. Without loss of generality, let $\mu_1 > \mu_2 \geq \ldots \geq \mu_k$, meaning that System 1 is the best of interest to select.

3.1 Key Idea

Recall that there are two issues to address when designing a sequential procedure: (1) justifying a Brownian approximation to the partial sum process of the differences; (2) finding a proper continuation region. In Section 2 we have addressed the second issue and the continuation region found there can be used in general settings as well. Without the normal assumptions of the samples, the Brownian approximation is not trivially valid. It appears that the difficulty in the general settings is to tackle the first issue.

The first issue is tackled using the following lemma. This lemma provides a justification of approximating the partial sum processes of differences by the Brownian motion processes. The lemma is known as Donsker's Functional Central Limit Theorem (see Billingsley 1968). Whitt (1970) generalizes this result from $D[0,1]$ to $D[0,\infty)$, where D is defined as Skorohod space.

Lemma 4 (Functional Central Limit Theorem) Suppose that $\{X_n : n\geq 1\}$ is a sequence of i.i.d. random variables with mean μ and variance $\sigma^2<\infty$. Let $S_n(t)$ denote the normalized partial sum process, i.e., $S_n(t) = n^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor}(X_i - nt\mu)/\sigma$. Then, $S_n(t)\Rightarrow B(t)$, in $D([0,\infty),\mathbb{R})$, as $n\to\infty$.

3.2 Procedure

In this subsection, a sequential IZ-free procedure for the general settings is proposed as follows .

Procedure 2

Setup. Select the PCS $1-\alpha (0<\alpha<1-1/k)$, and a common first-stage sample size $n_0 \geq 2$. Let $a=-2\log(2\alpha/(k-1))$.

Initialization. Let $I=\{i=1,2,\ldots,k\}$ be the set of systems still in contention. Simulate n_0 observations $X_{i1}, X_{i2}, \ldots, X_{in_0}$ from System i for all $i=1,2,\ldots,k$ and set $n=n_0$.

Screening. For any pair of systems $i,j \in I (i \neq j)$, calculate the sample variances of their difference, $S_{ij}^2(n) = (n-1)^{-1} \sum_{r=1}^n [X_{ir} - X_{jr} - (\bar{X}_i(n) - \bar{X}_j(n))]^2$. Define $I^{\text{old}} = I$ and

$$I = I^{\text{old}} \setminus \left\{ i \in I^{\text{old}} : Z_{ij}(n) < -g_{ij}(n) \text{ for some other } j \in I^{\text{old}} \right\},$$

where $Z_{ij}(n) = \sum_{r=1}^n (X_{ir} - X_{jr})$ and $g_{ij}(n) = \sqrt{S_{ij}^2(n)[a+\log(n+1)](n+1)}$.

Stopping Rule. If $|I|=1$, then stop and select the system whose index is in I as the best. Otherwise, take one additional observation from each Systems i for $i \in I$, set $n=n+1$ and go to **Screening**.

Based on the assumptions, variances of all simulated systems are unknown at the beginning of the procedure. In this procedure, variance estimators are updated sequentially as more samples are collected. This enables us to establish the statistical validity using variance estimators with strong consistency. Like Procedure 1, Procedure 2 approximates the partial sum processes of differences by a Brownian motion process and constructs a continuation region to determine the first exit time of the Brownian motion.

3.3 Statistical Validity

In this subsection, we establish the statistical validity of Procedure 2 in the asymptotic regime. To achieve this target, we need the following lemmas. Under some mild conditions, the following lemma states that the first exit times go to infinity as PCS goes to 1(i.e., $\alpha \to 0$). Based on this lemma, we can easily construct the strong consistency of the updated variance estimators.

Lemma 5 Let $Z(t)$ be a stochastic process with $\mathbb{E}[Z^2(t)] < \infty$ for all fixed t. Denote T_α as the stopping time $T_\alpha = \min\{t : |Z(t)| > g_\alpha(t)\}$, where $g_\alpha(t) \to \infty$ as $\alpha \to 0$, for any fixed t. Then $T_\alpha \to \infty$, with probability 1, as $\alpha \to 0$.

The following lemma provides the probability of a Brownian motion with no drift exiting from the continuation region used in the procedure.

Lemma 6 Let $T = \inf\{t : |B(t)| \geq g(t)\}$. Then $\mathbb{P}(T<\infty) \doteq e^{-a/2}$ where $g(t) = \sqrt{[a+\log(t+c)](t+c)}$.

Proof: This lemma can be verified based on Theorem 3.1 of Jennen and Lerche (1981).

Combining Lemmas 5 and 6 with Lemmas 2 and 3, we establish the asymptotic statistical validity of Procedure 2 and summarize it as the following theorem.

Theorem 4 Suppose that $X_i, i=1,2,\ldots,k$ are independent random variables with unknown means μ_i and unknown variances σ_i^2. Assume that $\sigma_i^2 < \infty$ for all $i=1,2,\ldots,k$ and $\mu_1 > \mu_2 \geq \ldots \geq \mu_k$. Then, Procedure 2 selects System 1 with $\limsup_{\alpha\to 0} \mathbb{P}(\text{ICS})/\alpha < 1$.

Proof: To show that $\limsup_{\alpha\to 0} \mathbb{P}(\text{ICS})/\alpha < 1$, it is equivalent to show that for any decreasing sequence $\{\alpha_m : m=1,2,\ldots\}$ with limit zero, such that $\limsup_{m\to\infty} \mathbb{P}(\text{ICS})/\alpha_m < 1$. By the definition of a, we also obtain an increasing sequence of $\{a_m : m=1,2,\ldots\}$ going to the infinity.

An incorrect-selection event happens when System 1 is eliminated by some other system before the termination of the procedure. To prove the statistical validity, we view each incorrect selection event separately. For any $i=2,\ldots,k$, we denote ICS_i as the incorrect selection event that System 1 is eliminated by System i. In particular, for a fixed m, $\mathbb{P}(\text{ICS}_\text{i}) = \mathbb{P}\left(Z_{1i}(T_m) < -\sqrt{S_{1i}^2(T_m)[a_m + \log(T_m+1)](T_m+1)} \right)$. where $T_m = \min\left\{ n : |Z_{1i}(n)| > \sqrt{S_{1i}^2[a_m+\log(n+1)](n+1)} \right\}$, $\sigma_{1i}^2 = \text{Var}[X_1 - X_i] = \sigma_1^2 + \sigma_i^2$. Noticing that $Z_{1i}(n) = \sum_{j=1}^n (X_{1j} - X_{ij})$ and $\mathbb{E}[X_{1j} - X_{ij}] = \mu_1 - \mu_i > 0$, we have $\mathbb{P}(T_m < \infty) = 1$ by the law of iterated logarithm. In other words, for the fixed m and $\varepsilon_m = \exp(-a_m^2)$, there exists some M_m such that $\mathbb{P}(T_m > M_m) \leq \varepsilon_m$. Let $M_m \to \infty$ as $m \to \infty$. Then,

$$\begin{aligned}\mathbb{P}(\mathrm{ICS_i}) &\leq \mathbb{P}\left(Z_{1i}(T_m) < -\sqrt{S_{1i}^2(T_m)(a_m + \log(T_m+1))(T_m+1)}, T_m \leq M_m\right) + \mathbb{P}(T_m > M_m)\\ &\leq \mathbb{P}\left(Z_{1i}(T_m^{(1)}) - T_m^{(1)}(\mu_1 - \mu_i) < -\sqrt{S_{1i}^2(T_m^{(1)})(a_m + \log(T_m^{(1)}+1))(T_m^{(1)}+1)}\right) + \varepsilon_m,\end{aligned}$$

where $T_m^{(1)} = M_m \bigwedge \min\left\{n : n \in \mathbf{Z}^+, |Z_{1i}(n) - n(\mu_1 - \mu_i)| \geq \sqrt{S_{1i}^2(n)(a_m + \log(n+1))(n+1)}\right\}$. (Here $x \bigwedge y = \min\{x,y\}$). The second inequality holds because it is easier for a process with a smaller drift to leave the continuation region from below. Define

$$C(t, M_m) = \frac{\sum_{j=1}^{\lfloor M_m t \rfloor}(X_{1j} - X_{ij}) - M_m t(\mu_1 - \mu_i)}{\sigma_{1i}\sqrt{M_m}t}, \text{ for } t \in [0,1]. \tag{1}$$

By Lemma 4, we have $C(t, M_m) \Rightarrow B(t)$ in $D[0,1]$ as $M_m \to \infty$. Hence,

$$\begin{aligned}&\mathbb{P}\left(Z_{1i}(T_m^{(1)}) - T_m^{(1)}(\mu_1 - \mu_i) < -\sqrt{S_{1i}^2(T_m^{(1)})[a_m + \log(T_m^{(1)}+1)](T_m^{(1)}+1)}\right)\\ =&\mathbb{P}\left(C(T_m^{(2)}, M_m) \leq -\sqrt{S_{1i}^2(M_m T_m^{(2)})/\sigma_{1i}^2[a_m + \log(M_m T_m^{(2)}+1)](M_m T_m^{(2)}+1)/M_m}\right),\end{aligned}$$

where $T_m^{(2)} = T_m^{(1)}/M_m$. For fixed M_m, $C(T_m^{(2)}, M_m)$ corresponds to the right-hand limit of a discontinuous point of $C(\cdot, M_m)$. Let $T_m^{(3)}$ define a stopping time corresponding to the continuous process $C(\cdot, M_m)$ as $T_m^{(3)} = 1 \bigwedge \inf\left\{t : |C(t, M_m)| > \sqrt{S_{1i}^2(M_m t)/\sigma_{1i}^2\,[a + \log(M_m t+1)]\,(M_m t+1)/M_m}\right\}$. Besides, we can show that $T_m^{(2)} \to T_m^{(3)}, a.s.$, as $M_m \to \infty$. Hence, in the limit, it is equivalent to consider

$$\mathbb{P}\left(C(T_m^{(3)}, M_m) < -\sqrt{S_{1i}^2(M_m T_m^{(3)})/\sigma_{1i}^2[a_m + \log(M_m T_m^{(3)}+1)](M_m T_m^{(3)}+1)/M_m}\right). \tag{2}$$

Notice that $M_m \to \infty, a.s.$, as $\alpha_k \to 0$, which implies that $S_{1i}^2(M_m T_m^{(3)}) \to \sigma_{1i}^2, a.s.$, as $\alpha \to 0$. Hence, based on Slutsky's theorem (see, e.g., Durrett 2010) and Lemma 1, using a similar proof of Kim, Nelson, and Wilson (2005), we can derive that for the sufficiently large m,

$$\begin{aligned}(2) &\leq \mathbb{P}\left(B(\tilde{T}_m) < -\sqrt{[a_m + \log(M_m\tilde{T}_m+1)](M_m\tilde{T}_m+1)/M_m}\right)\\ &\leq \mathbb{P}\left(B(\tilde{T}_m^*) < -\sqrt{[a_m + \log(M_m\tilde{T}_m^*+1)](M_m\tilde{T}_m^*+1)/M_m}\right),\end{aligned}$$

where $\tilde{T}_m^* = \inf\left\{t : |B(t)| > \sqrt{[a_m + \log(M_m t+1)]\,(M_m t+1)/M_m}\right\}$ and $\tilde{T}_m^* = 1 \bigwedge \tilde{T}_m$. A rigorous version of this proof is omitted due to page limit. Then, $\mathbb{P}\left(B(\tilde{T}_m) < -\sqrt{[a_m + \log(M_m\tilde{T}_m+1)](M_m\tilde{T}_m+1)/M_m}\right) \doteq \exp(-a_m/2 - \log M_m/2)$, by Lemma 2. Since $M_m \to \infty$, without the loss of generality, we can assume that $M_m \geq 1$. Then, $\exp(-a_m/2 - \log M_m/2) \leq \frac{1}{2}\exp(-a_m/2) = \alpha_m/(k-1)$. Hence, we have that $\lim_{m\to\infty}\mathbb{P}(\mathrm{ICS_i})/(\alpha_m/(k-1)) \leq 1$. Furthermore, $\lim_{m\to\infty}\mathbb{P}(\mathrm{ICS})/\alpha_m \leq \lim_{m\to\infty}\sum_{i=2}^k \mathbb{P}(\mathrm{ICS_i})/\alpha_m \leq 1$. Therefore, $\limsup_{\alpha\to 0}\mathbb{P}(\mathrm{ICS_i})/\alpha \leq 1$.

4 ASYMPTOTIC SAMPLE SIZE

Other than the probability of correct selection, another measure of the goodness of a selection-of-the-best procedure is the average sample size required to select the best system. In this section, we put our attention on Procedure 1 because the analysis can be easily extended to other procedures we design in this paper. We study the asymptotic sample size of Procedure 1 and compare it with that of the *KN* procedure with known variances.

4.1 Analytic Expression for Asymptotic Sample Size

Consider a pairwise comparison between Systems 1 and System i. Denote the difference of their means as $\mu_1 - \mu_i$. As the sample size goes to infinity, the partial sum process of their difference behaves more and more like a line with the slope $\mu_1 - \mu_i$. Hence, a heuristic method to calculate the average sample size is to find the intersection point between the boundary and the line, see Figure 2. The method is known as the mean path approximation. Perng (1969) provides a rigorous proof of this heuristic method for Paulson's procedure and the proof can be easily extended to the sequential procedures with a triangular continuation region (such as the *KN* procedure). However, a rigorous proof has not been given so far for sequential procedures with a continuation region, such as the one used in our procedure. In this subsection, we show this heuristic method is also valid for our procedure under mild conditions.

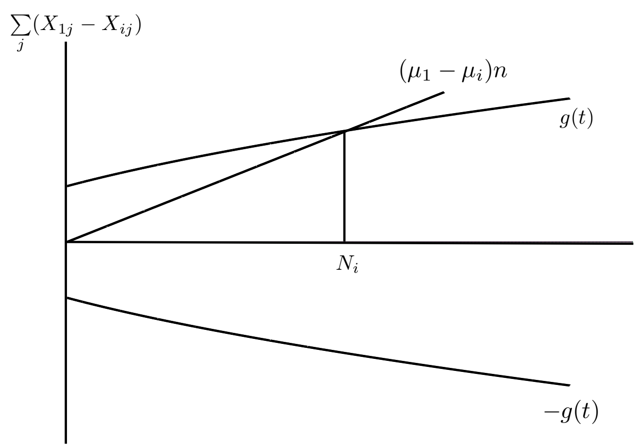

Figure 2: Heuristic Mean Path Approximation Method

Theorem 5 Suppose that $X_1, X_2, \ldots, X_k$ are independent random variables with unknown means μ_i and known variances σ_i^2, for all $i = 1, 2, \ldots, k$. Define N_i as the sample size required to distinguish μ_1 and μ_i, for $i \neq 1$. If $\mu_1 > \mu_2 \geq \ldots \geq \mu_k$, then, in Procedure 1, $\mathbb{E}[N_i] \doteq \lambda_i$, where λ_i is the solution to $\sqrt{(\sigma_1^2 + \sigma_i^2)[a + \log(\lambda_i + 1)](\lambda_i + 1)} = (\mu_1 - \mu_i)\lambda_i$, for all $i = 2, 3, \ldots, k$.

Proof: To verify the theorem, it's equivalent to prove the following statements: (1) $\limsup_{\alpha \to 0} \mathbb{E}[N_i]/\lambda_i \leq 1$; (2) $\liminf_{\alpha \to 0} \mathbb{E}[N_i]/\lambda_i \geq 1$, for all $i = 2, 3, \ldots, k$. The proof can be summarized as follows. To prove the first statement, we construct an upper linear support for our boundary and show its average sample size is asymptotically λ_i under this linear boundary. Following this, we show our procedure has a smaller asymptotic average sample size than the one using the linear boundary. As for the second one, we show that it holds for a subset of the event space, in which the sample mean converges to true mean uniformly.

4.2 Comparisons between Our Procedure and the *KN* Procedure

As a representative sequential IZ procedure, the *KN* procedure is thoroughly studied in the literature and widely used in practice. Hence, in this subsection, we compare our procedure with it in terms of the asymptotic sample sizes required to select the best system.

Notice that sequential procedures often break the selection problem into a group of pairwise comparisons. As a building block, we first compare samples sizes for the case of two systems. Let Δ as the difference of means between these two systems and the variance of their difference is 1. It is easy to calculate the asymptotic sample size required by our procedure based on Theorem 5. Given any specified IZ parameter δ, the asymptotic sample size of the *KN* procedure can be also evaluated.

To compare their performances in terms of asymptotic sample size, we consider a varying IZ parameter δ. The larger Δ/δ is, the more conservative the *KN* procedure is. In particular, the IZ parameter is accurately specified if $\Delta/\delta = 1$. In Figure 3, we plot the ratios of the sample sizes of the *KN* procedure and our procedure for different values of Δ/δ.

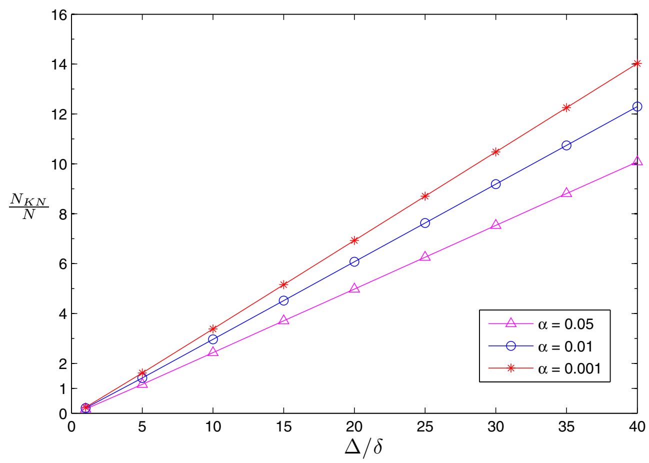

Figure 3: Comparsions of sample sizes using the *KN* procedures and our procedure when IZ parameter set in the *KN* procedure varies.

From Figure 3, we find that our procedure needs about five times the samples of the *KN* procedure when the IZ parameter is accurately specified. However, when Δ/δ is above 5, our procedure begins to out-perform the *KN* procedure. Moreover, as the specified IZ parameter δ becomes more and more conservative, our procedure may require fewer and fewer samples than the *KN* procedure.

When the number of alternative systems is large, the advantage of the *KN* procedure diminishes. In this case, there are often many systems whose means are significantly smaller than that of the best system. Therefore, even if the IZ parameter δ is specified accurately, many of the pairwise comparisons may have a large value of Δ/δ, thus diminishing the advantage of the *KN* procedure.

5 Numerical Experiments

In this section, we test the performance of our procedure through numerical experiments. For simplicity, we consider a problem where the samples are normally distributed. We compare our procedure with the *KN*++ procedure (Kim and Nelson 2006) because both procedures update the variance estimators sequentially.

We consider selection-of-the-best problems with different numbers of alternative systems where $\mu_i = 1 - 0.5(i-1)$ and $\sigma_i^2 = 10$, for all $i = 1, 2, \ldots, k$. We report the estimated PCS and the average sample sizes for $k = 20, 50, 100, 500$ and $(\mu_1 - \mu_2)/\delta = 1, 2, 4, 8$ in Table 1.

From Table 1, we have two findings. First, for any fixed number of systems (see each row in Table 1), our IZ-free procedure demands fewer samples than the *KN*++ procedure, when the IZ parameter is less than half of $\mu_1 - \mu_2$. Furthermore, the differences between the average sample sizes increase as $(\mu_1 - \mu_2)/\delta$ increases. Second, from the third and fourth columns, we find that, although our procedure needs more samples than the *KN*++ procedure with an accurately specified IZ parameter, the differences of average sample sizes tend to narrow down as the number of alternative systems k increases. We also repeat this experiments for other configurations of variances and the results are similar, see Table 2 for an instance.

Table 1: Comparison of our procedure and the *KN*++ procedure to select the largest mean from k normal variables with $(\mu_1, \mu_2, \mu_3, \ldots, \mu_k) = (1, 0.5, 0, \ldots, 0)$ and variances 10, and the PCS is set as 0.95.

			$(\mu_1 - \mu_2)/\delta$			
	k	IZ-free	1	2	4	8
Estimated PCS	20	0.995	0.998	1.000	1.000	1.000
Sample Size		3.29E+03	1.38E+03	3.24E+03	6.98E+03	1.47E+04
Estimated PCS	50	0.999	0.999	1.000	1.000	1.000
Sample Size		3.95E+03	1.99E+03	4.45E+03	9.79E+03	2.07E+04
Estimated PCS	100	0.999	1.000	1.000	1.000	1.000
Sample Size		4.77E+03	2.71E+03	5.52E+03	1.18E+04	2.51E+04
Estimated PCS	500	1.000	1.000	1.000	1.000	1.000
Sample Size		9.52E+03	7.24E+03	1.08E+04	1.88E+04	3.62E+04

Table 2: Comparison of out procedure and *KN*++ procedure to select the largest mean from 100 normal variables with $(\mu_1, \mu_2, \mu_3, \ldots, \mu_{100}) = (1, 0.5, 0, \ldots, 0)$ and different configurations of variances, and the PCS is set as 0.95.

			$(\mu_1 - \mu_2)/\delta$			
	Variances	IZ-free	1	2	4	8
Estimated PCS	Equal	0.999	1.000	1.000	1.000	1.000
Sample Size		4.77E+03	2.71E+03	5.52E+03	1.18E+04	2.51E+04
Estimated PCS	Increasing	0.998	1.000	1.000	1.000	1.000
Sample Size		5.16E+03	3.65E+03	8.35E+03	1.83E+04	3.82E+04
Estimated PCS	Decreasing	1.000	0.999	1.000	1.000	1.000
Sample Size		4.46E+03	2.40E+03	4.64E+03	9.35E+03	1.96E+04

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