Kernel Estimation of Quantile Sensitivities

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Abstract: Quantiles, also known as value-at-risks in the financial industry, are important measures of random performances. Quantile sensitivities provide information on how changes in input parameters affect output quantiles. They are very useful in risk management. In this article, we study the estimation of quantile sensitivities using stochastic simulation. We propose a kernel estimator and prove that it is consistent and asymptotically normally distributed for outputs from both terminating and steady-state simulations. The theoretical analysis and numerical experiments both show that the kernel estimator is more efficient than the batching estimator of Hong [9]. Naval Research Logistics 56: 511–525, 2009

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1. INTRODUCTION

Quantiles are also known as value-at-risks (VaRs) in the financial industry. They are widely used as measures of random performances. In the financial industry, for instance, VaRs are used to measure the default risks of banks and they have been used by various regulatory agencies to regulate the capital adequacy of banks (Jorion [1]). In the service industry, for instance, the quantiles of the time taken to respond to emergency requests and to transport patients to the hospital are used to measure the service qualities of an out-of-hospital system (Austin and Schull [2]).

In practice, random performance often depends on many parameters. The quantile also depends on these parameters. Quantile sensitivities are first-order derivatives of the quantile with respect to these parameters. They provide information on how changes in the parameters affect the quantile value. Quantile sensitivities are useful in quantifying model adequacy. For instance, models of financial losses often have parameters that need to be estimated and are subject to estimation errors. If the VaR sensitivity with respect to a parameter is high, then the loss model may not be adequate since a small error in the estimation of the parameter may cause the VaR to change significantly. Quantile sensitivities are also useful in optimizing quantile performances. For instance, portfolio optimization problems can often be modeled as minimizing portfolio risk subject to a certain level of expected returns. When the risk is measured by VaR, then VaR sensitivities can be used to solve the optimization problems efficiently.

Estimating sensitivities of expectations has been studied extensively in the simulation literature. Typical methods include perturbation analysis, the likelihood ratio (or score function) method and the weak derivative method. Readers are referred to Fu [3] and L'Ecuyer [4] for comprehensive reviews. Estimating VaR sensitivities with respect to portfolio allocation parameters has been studied in the finance literature. Gourieroux et al. [5] used a kernel-based recursive algorithm to approximate VaR and its sensitivities with respect to the portfolio allocation parameters. Recently, for the capital allocation problem for credit portfolios, Tasche [6] and Epperlein and Smillie [7] applied kernel methods to estimate risk contributions of VaR, and Glasserman [8] considered the importance sampling issue for estimating risk contributions of VaR and expected shortfall, where the risk contributions are the sensitivities of the VaR or the expected shortfalls associated with individual obligors or transactions. The estimation of quantile sensitivities for general functions was recently studied by Hong [9], who showed that a quantile sensitivity can be written as a conditional expectation. He then proposed a batching estimator and proved its consistency and asymptotic normality. Hong and Liu [10] studied the sensitivity estimation of conditional VaR (CVaR), which is another widely used risk measure. They showed that the conditional-expectation form of the quantile sensitivity of Hong [9] can also be derived by differentiating the CVaR sensitivity.

Additional Supporting Information may be found in the online version of this article.

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In this article, we propose a kernel estimator for estimating quantile sensitivities. We prove the consistency and asymptotic normality of the estimator for observations from both terminating simulations and steady-state simulations. In the case of terminating simulations, we consider that the simulation observations are independent and identically distributed (i.i.d.), and in the case of steady-state simulations, we consider that simulation observations are ϕ-mixing. These setups are also used by Hong and Liu [11] to study the estimation of probability sensitivities. However, the asymptotic analysis of our kernel estimator is more challenging than that of the probability sensitivity of [11] for two reasons. First, our estimator is a ratio estimator. Second, our estimator uses a quantile estimator that depends on all the simulation observations.

In short, the contributions of this article are twofold. First, we propose a kernel estimator for estimating quantile sensitivities that converges faster than the batching estimator proposed by Hong [9]. Specifically, the optimal rate of convergence of the kernel estimator is $n^{-2/5}$ while that of the batching estimator is $n^{-1/3}$. Second, we show that the kernel estimator is consistent and follows a central limit theorem for both terminating and steady-state simulations.

The rest of the article is organized as follows: Section 2 describes the background and proposes a kernel estimator of the quantile sensitivity. The consistency and asymptotic normality of the estimator for terminating and steady-state simulations are considered in Sections 3 and 4, respectively. Section 5 discusses how to choose the bandwidth for the estimator. We compare the kernel estimator and the batching estimator through several numerical examples in Section 6, followed by conclusions in Section 7. Some lengthy proofs are provided in the Appendix.

2. BACKGROUND

Let $L(\theta)$ denote the random output of a simulation model, where θ is the parameter that we are interested in. Throughout the article, we suppose that θ is a scalar. When it is not, we may consider each dimension as a scalar while holding all other dimensions fixed. We further assume that $\theta \in \Theta$, and Θ is an open set in $\mathbb{R}$. In queueing systems, for instance, θ may be the arrival rate or service rate and $L(\theta)$ may be the sojourn time, that is, the time that a customer spends in the system. In financial models, for instance, θ may be the interest rate or exchange rate, and $L(\theta)$ may be the random loss of a portfolio of many financial securities.

For any $\theta \in \Theta$, let $q_\alpha(\theta)$ denote the α-quantile of $L(\theta)$. When $L(\theta)$ is a continuous random variable, $q_\alpha(\theta)$ satisfies $\Pr\{L(\theta) \le q_\alpha(\theta)\} = \alpha$. Hong [9] proves that the quantile sensitivity, $q'_\alpha(\theta)$, can be written as a conditional expectation. He makes the following assumptions.

ASSUMPTION 1: The pathwise derivative, $D(\theta) = L'(\theta)$, exists with probability 1 (w.p.1) for any $\theta \in \Theta$, and there exists a random variable Y, with $E[Y] < \infty$, such that $|L(\theta_2) - L(\theta_1)| \le Y|\theta_2 - \theta_1|$ for any $\theta_1, \theta_2 \in \Theta$.

ASSUMPTION 2: For any $\theta \in \Theta$, $L(\theta)$ has a continuous density function in the neighborhood of $q_\alpha(\theta)$, and $q_\alpha(\theta)$ is differentiable with respect to θ.

ASSUMPTION 3: For any $\theta \in \Theta$, $E[D(\theta)|L(\theta) = t]$ is continuous at $t = q_\alpha(\theta)$.

Then, Hong [9] proves the following lemma, which serves as the basis of our estimator.

LEMMA 1: Suppose that Assumptions 1–3 are satisfied. Then, $q'_\alpha(\theta) = E[D(\theta)|L(\theta) = q_\alpha(\theta)]$.

Let $\{(L_i(\theta), D_i(\theta)), i = 1, 2, \ldots, n\}$ be the observations of $(L(\theta), D(\theta))$ from the simulation. They may be obtained from either terminating or steady-state simulations. For many simulation models, the pathwise derivative, $D(\theta)$, can be obtained with only a little additional computation effort from the simulation. If the price of an asset follows a diffusion process, its pathwise derivative with respect to the interest rate can be calculated easily based on the sample path itself (Broadie and Glasserman [12]). If we are interested in the sojourn time of a queueing system, the pathwise derivative with respect to an arrival rate or a service rate can be calculated using infinitesimal perturbation analysis (Glasserman [13]).

To simplify the notation, we let (L, D) and (L_i, D_i) denote $(L(\theta), D(\theta))$ and $(L_i(\theta), D_i(\theta))$, respectively, and we let q_α denote $q_\alpha(\theta)$, when there is no confusion. Throughout the article, we assume that (L, D) is a continuous bivariate random vector with a joint density $f(x, t)$. For a nonnegative integer m, we define

$$g_m(x) = \int_{-\infty}^{\infty} t^m f(x,t)dt \quad \text{and} \quad h_m(x) = \int_{-\infty}^{\infty} |t|^m f(x,t)dt.$$

By Lemma 1,

$$q'_\alpha(\theta) = E[D(\theta)|L(\theta) = q_\alpha] = \frac{g_1(q_\alpha)}{g_0(q_\alpha)}. \tag{1}$$

In this article, we propose a kernel estimator to estimate $q'_\alpha(\theta)$ using the observations $\{(L_i, D_i), i = 1, 2, \ldots, n\}$. Suppose that K is a bounded symmetric density such that $yK(y) \to 0$ as $|y| \to \infty$ and $\int_{-\infty}^{\infty} y^2 K(y)dy < \infty$. Then, K is a kernel on $\mathbb{R}$ (Bosq [14]). For instance, the standard normal density function is a kernel. The kernel method has been studied extensively in the area of nonparametric statistics, especially in density estimation and nonparametric regression analysis (see, for instance, Bosq [14], Li and Racine [15], and Pagan and Ullah [16] for recent overviews). Researchers have found that the kernel method has several advantages: it is generally easy to compute and robust; and it reaches the optimal convergence rate in terms of the quadratic error. The kernel method typically involves a bandwidth parameter δ_n, which satisfies $\delta_n \to 0$ and $n\delta_n \to \infty$ as $n \to \infty$. In this article, we make the following assumption on the kernel K.

ASSUMPTION 4: For $m = 0, 1, \ldots, 4$, $[K^{(m)}]^2$ is bounded and integrable, and $yK^{(m)}(y) \to 0$ as $|y| \to \infty$, where the superscript "(m)" denotes the m-th order derivative.

REMARK 1: This is a typical assumption on the kernel. It is satisfied when K is the standard normal density function.

Let $\hat{q}_\alpha^n = L_{\lceil n\alpha \rceil:n}$ be the estimator of q_α, where $L_{i:n}$ denotes the i-th order statistic from the n observations of L. Serfling [17] shows that $\hat{q}_\alpha^n \to q_\alpha$ w.p.1 and $\sqrt{n}(\hat{q}_\alpha^n - q_\alpha)$ converges to a normal distribution as $n \to \infty$ if L has a positive density in a neighborhood of q_α. Let

$$\bar{V}_n = \frac{\sum_{i=1}^n K\left(\frac{\hat{q}_\alpha^n - L_i}{\delta_n}\right) \cdot D_i}{\sum_{i=1}^n K\left(\frac{\hat{q}_\alpha^n - L_i}{\delta_n}\right)} \tag{2}$$

be our kernel estimator of $q'_\alpha(\theta)$. In the rest of the article, we show that $\bar{V}_n$ is a consistent and asymptotically normally distributed estimator of $q'_\alpha(\theta)$ for both terminating and steady-state simulations.

We further define

$$\bar{R}_n(y) = \frac{1}{n\delta_n} \sum_{i=1}^n K\left(\frac{y - L_i}{\delta_n}\right) \cdot D_i, \qquad \bar{Q}_n(y) = \frac{1}{n\delta_n} \sum_{i=1}^n K\left(\frac{y - L_i}{\delta_n}\right),$$

and let $\bar{V}_n(y) = \bar{R}_n(y)/\bar{Q}_n(y)$ for any $y \in \mathbb{R}$. Note that $\bar{V}_n = \bar{V}_n(\hat{q}_\alpha^n)$. It has been shown in the literature that $\bar{R}_n(y)$, $\bar{Q}_n(y)$, and $\bar{V}_n(y)$ are consistent and asymptotically normally distributed estimators of $g_1(y)$, $g_0(y)$, and $g_1(y)/g_0(y)$, respectively, for any fixed $y \in \mathbb{R}$ under appropriate conditions, when $(L_1, D_1), \ldots, (L_n, D_n)$ are i.i.d. observations (see, for instance, Schuster [18] and Watson [19]), or when $(L_1, D_1), \ldots, (L_n, D_n)$ are dependent observations (see, for instance, Roussas [20], Roussas and Tran [21], and Truong [22]). Therefore, when $y = q_\alpha$, $\bar{V}_n(q_\alpha)$ is a consistent and asymptotically normal distributed estimator of $q'_\alpha(\theta)$ by Eq. (1).

In our problem, however, $\bar{V}_n$ equals $\bar{V}_n(\hat{q}_\alpha^n)$ rather than $\bar{V}_n(q_\alpha)$. The quantity q_α is unknown, and it is estimated by $\hat{q}_\alpha^n$. Because $\hat{q}_\alpha^n$ links all observations, $(L_1, D_1), \ldots, (L_n, D_n)$, together, the standard analysis of the kernel method is no longer applicable to $\bar{V}_n = \bar{V}_n(\hat{q}_\alpha^n)$. In the rest of the paper, we solve this problem and show that $\bar{V}_n$ is still a consistent and asymptotically normally distributed estimator of $q'_\alpha(\theta)$ for both terminating and steady-state simulations.

3. TERMINATING SIMULATION

Suppose that $\{(L_i, D_i), i = 1, 2, \ldots, n\}$ are generated from terminating simulations. Then, they are independent and identically distributed (see, for instance, Law and Kelton [23]). In this section, we consider the consistency and asymptotic normality of $\bar{V}_n$ for these i.i.d. observations. To conduct the analysis, we need some regularity conditions, stated in the following assumption.

ASSUMPTION 5: The functions $g_0^{(4)}(t)$, $g_1^{(4)}(t)$ and $g_2(t)$ are continuous at $t = q_\alpha$ and $g_0(q_\alpha) > 0$. For $m = 0, 1, \ldots, 4$, $\int_{-\infty}^{\infty} |g_1^{(m)}(t)| dt < \infty$.

REMARK 2: Note that $g_0(t)$ is the density function of L, $g_1(t) = g_0(t)E[D|L = t]$ and $g_2(t) = g_0(t)E[D^2|L = t]$. Assumption 5 basically requires that L has a positive density at the point $t = q_\alpha$ and (L, D) has good mathematical properties in a neighborhood of $L = q_\alpha$.

3.1. Consistency

Recall that $\bar{V}_n = \bar{R}_n(\hat{q}_\alpha^n)/\bar{Q}_n(\hat{q}_\alpha^n)$ and $q'_\alpha(\theta) = g_1(q_\alpha)/g_0(q_\alpha)$. Then

$$\bar{V}_n - q'_\alpha(\theta) = \frac{1}{\bar{Q}_n(\hat{q}_\alpha^n) g_0(q_\alpha)} \times \left[g_0(q_\alpha)\left(\bar{R}_n(\hat{q}_\alpha^n) - g_1(q_\alpha)\right) - g_1(q_\alpha)\left(\bar{Q}_n(\hat{q}_\alpha^n) - g_0(q_\alpha)\right)\right]. \tag{3}$$

To prove the consistency of $\bar{V}_n$, i.e., $\bar{V}_n \xrightarrow{P} q'_\alpha(\theta)$ where "$\xrightarrow{P}$" denotes "*convergence in probability*," we need to show that $\bar{R}_n(\hat{q}_\alpha^n) \xrightarrow{P} g_1(q_\alpha)$ and $\bar{Q}_n(\hat{q}_\alpha^n) \xrightarrow{P} g_0(q_\alpha)$. Because the consistency of $\bar{R}_n(q_\alpha)$ and $\bar{Q}_n(q_\alpha)$ has already been established in the literature, it suffices to prove that both $\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)$ and $\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)$ converge to 0 in probability.

We summarize the consistency of $\bar{R}_n(q_\alpha)$ and $\bar{Q}_n(q_\alpha)$ in the following lemma. It has been proved under various sets of conditions (see, e.g., Bosq [14], Li and Racine [15] and Pagan and Ullah [16]). Because the set of conditions in our lemma is slightly different from the ones in the literature, we provide a proof of the lemma in Liu and Hong [24] in the supplementary materials for this paper.

LEMMA 2: Suppose that Assumptions 4–5 are satisfied and $E(D^2) < \infty$. If $\delta_n \to 0$ and $n\delta_n \to \infty$ as $n \to \infty$, then $\bar{R}_n(q_\alpha) \xrightarrow{P} g_1(q_\alpha)$ and $\bar{Q}_n(q_\alpha) \xrightarrow{P} g_0(q_\alpha)$.

Lemma 2 shows that $\bar{V}_n(q_\alpha) = \bar{R}_n(q_\alpha)/\bar{Q}_n(q_\alpha)$ is a consistent estimator of $q'_\alpha(\theta)$. However, it is not directly applicable to $\bar{V}_n = \bar{V}_n(\hat{q}^n_\alpha)$, where $\hat{q}^n_\alpha$ is a strongly consistent estimator of q_α. To show the consistency of $\bar{V}_n$, we need the following lemma that is proved in the Appendix.

LEMMA 3: Suppose that Assumptions 4–5 are satisfied. If $\delta_n \to 0$ as $n \to \infty$, then for $m = 0, 1, \ldots, 4$,

$$\lim_{n\to\infty} E\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right) D_i\right] = g_1^{(m)}(q_\alpha),$$

$$\lim_{n\to\infty} n\delta_n^{2m+1}\text{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right) D_i\right] = \sigma_m^2,$$

where

$$\sigma_m^2 = g_2(q_\alpha)\int_{-\infty}^{\infty}[K^{(m)}(t)]^2 dt.$$

Now, we are ready to prove the consistency of $\bar{V}_n$.

THEOREM 1: Suppose that Assumptions 1–5 are satisfied and $E(D^2) < \infty, \delta_n \to 0$ as $n \to \infty$, and $\sup_n (n\delta_n^{5/2})^{-1} < \infty$. Then, $\bar{V}_n$ is a consistent estimator of $q'_\alpha(\theta)$.

PROOF: By Eq. (3) and Lemma 2, it suffices to show that

$$\bar{R}_n(\hat{q}^n_\alpha) - \bar{R}_n(q_\alpha) \xrightarrow{P} 0 \quad \text{and} \quad \bar{Q}_n(\hat{q}^n_\alpha) - \bar{Q}_n(q_\alpha) \xrightarrow{P} 0.$$

For simplicity of the notation, we let $K^{(m)}_{n,i}$ denote $K^{(m)}(\frac{q_\alpha - L_i}{\delta_n})$ for $m = 1, 2, 3, 4$.

By Assumption 4, $K^{(4)}(\cdot)$ is bounded. Then, by Taylor's expansion,

$$\begin{aligned}
&\bar{R}_n(\hat{q}^n_\alpha) - \bar{R}_n(q_\alpha) \\
&= \frac{1}{n\delta_n}\sum_{i=1}^n K\left(\frac{\hat{q}^n_\alpha - L_i}{\delta_n}\right)\cdot D_i - \frac{1}{n\delta_n}\sum_{i=1}^n K\left(\frac{q_\alpha - L_i}{\delta_n}\right)\cdot D_i \\
&\quad = \sum_{m=1}^4 \frac{1}{m!}\left[\sqrt{n}(\hat{q}^n_\alpha - q_\alpha)\right]^m \frac{1}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i \\
&\qquad + o_p\left(\left(\frac{\hat{q}^n_\alpha - q_\alpha}{\delta_n}\right)^4 \frac{1}{n\delta_n}\sum_{i=1}^n |D_i|\right), \qquad (4)
\end{aligned}$$

where $o_p(\cdot)$ is a random sequence that satisfies $A_n/B_n \xrightarrow{P} 0$ if $A_n = o_p(B_n)$.

By Lemma 3, for $m = 1, 2, 3, 4$,

$$\begin{aligned}
&\lim_{n\to\infty} E\left[\frac{1}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i\right] \\
&\qquad = \lim_{n\to\infty}\frac{1}{n^{m/2}} E\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i\right] = 0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\text{Var}\left[\frac{1}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i\right] \\
&= \lim_{n\to\infty}\frac{1}{n\delta_n(n\delta_n^2)^m}\cdot n\delta_n^{2m+1}\cdot\text{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i\right] = 0,
\end{aligned}$$

where the last equation holds since $n\delta_n \to \infty$ and $n\delta_n^2 \to \infty$ as $n \to \infty$ when $\sup_n (n\delta_n^{5/2})^{-1} < \infty$. Hence, by Chebyshev's inequality (Durrett [25]), for $m = 1, 2, 3, 4$,

$$\frac{1}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K^{(m)}_{n,i}\cdot D_i \xrightarrow{P} 0.$$

Note that $\sqrt{n}(\hat{q}^n_\alpha - q_\alpha)$ converges to a normal distribution (Serfling [17]). Then, by the continuous mapping theorem (Durrett [25]), $[\sqrt{n}(\hat{q}^n_\alpha - q_\alpha)]^m$ converges to some distribution. Therefore, the first term of Eq. (4) converges to 0 in probability by Slutsky's theorem (Serfling [17]).

Furthermore, by the strong law of large numbers, $\frac{1}{n}\sum_{i=1}^n |D_i| \xrightarrow{P} E(|D|)$. Recall that $n^2(\hat{q}^n_\alpha - q_\alpha)^4$ converges to some distribution and $\sup_n (n\delta_n^{5/2})^{-1} < \infty$. Then, by Slutsky's theorem, the second term of Eq. (4) also converges to 0 in probability. Therefore, $\bar{R}_n(\hat{q}^n_\alpha) - \bar{R}_n(q_\alpha) \xrightarrow{P} 0$.

Similarly, we can show that $\bar{Q}_n(\hat{q}^n_\alpha) - \bar{Q}_n(q_\alpha) \xrightarrow{P} 0$. This concludes the proof. □

3.2. Asymptotic Normality

In this subsection, we consider the asymptotic normality of $\bar{V}_n$, which can help to compare the asymptotic efficiencies of our kernel estimator and Hong's [9] batching estimator. It can also help to construct confidence intervals for $q'_\alpha(\theta)$. For discussions on confidence intervals and their use in simulation estimation, readers are referred to Law and Kelton [23].

The idea behind the proof of the asymptotic normality of $\bar{V}_n$ is quite similar to that of the consistency. Because existing results in nonparametric statistics have shown that $\sqrt{n\delta_n}[\bar{V}_n(q_\alpha) - q'_\alpha(\theta)]$ converges to a normal distribution, by Slutsky's theorem (Serfling [17]), it suffices to show that $\sqrt{n\delta_n}[\bar{V}_n - \bar{V}_n(q_\alpha)] \xrightarrow{P} 0$.

We summarize the asymptotic normality of $\bar{V}_n(q_\alpha)$ in the following lemma. Similar to Lemma 2, Lemma 4 is also a known result (see, e.g., Bosq [14], Li and Racine [15] and Pagan and Ullah [16]). We include our proof of the lemma in Liu and Hong [24] for completeness.

LEMMA 4: Suppose that Assumptions 1–5 are satisfied and $E(|D|^{2+\gamma}) < \infty$ for some $\gamma > 0$ and $h_{2+\gamma}(t)$ is continuous at $t = q_\alpha$. If $\delta_n \to 0$, $n\delta_n \to \infty$ and $n\delta_n^5 \to c$ as $n \to \infty$ with $c \geq 0$, then

$$\sqrt{n\delta_n}\left[\bar{V}_n(q_\alpha) - q'_\alpha(\theta)\right] \Rightarrow \mu_\infty + \sigma_\infty \cdot N(0,1),$$

where "$\Rightarrow$" denotes "convergence in distribution," $N(0,1)$ denotes the standard normal distribution, and

$$\mu_\infty = \frac{c}{g_0^2(q_\alpha)}\left[g_0(q_\alpha)g_1''(q_\alpha) - g_1(q_\alpha)g_0''(q_\alpha)\right]\int_{-\infty}^{\infty} t^2K(t)dt,$$

$$\sigma_\infty^2 = \frac{g_0(q_\alpha)g_2(q_\alpha) - g_1^2(q_\alpha)}{g_0^3(q_\alpha)}\int_{-\infty}^{\infty} K^2(t)dt.$$

To analyze the asymptotic normality of $\bar{V}_n = \bar{V}_n(\hat{q}_\alpha^n)$, we also need the following lemma that is proved in the Appendix.

LEMMA 5: Suppose that Assumptions 4–5 are satisfied. If $\delta_n \to 0$ as $n \to \infty$ and $\sup_n (n\delta_n^3)^{-1} < \infty$, then $\sqrt{n\delta_n}[\bar{V}_n - \bar{V}_n(q_\alpha)] \xrightarrow{P} 0$.

Combining Lemmas 4 and 5 together, we immediately obtain the following theorem on the asymptotic normality of $\bar{V}_n$.

THEOREM 2: Suppose that Assumptions 1–5 are satisfied and $E(|D|^{2+\gamma}) < \infty$ for some $\gamma > 0$ and $h_{2+\gamma}(t)$ is continuous at $t = q_\alpha$. If $\delta_n \to 0$ and $n\delta_n^5 \to c$ with $c \geq 0$ as $n \to \infty$ and $\sup_n (n\delta_n^3)^{-1} < \infty$, then

$$\sqrt{n\delta_n}\left[\bar{V}_n - q'_\alpha(\theta)\right] \Rightarrow \mu_\infty + \sigma_\infty \cdot N(0,1),$$

where μ_∞ and σ_∞ are defined in Lemma 4.

Theorem 2 shows that the rate of convergence of $\bar{V}_n$ is $(n\delta_n)^{-1/2}$. It is slower than the typical $n^{-1/2}$ rate of convergence of a Monte Carlo estimator. This is because $q'_\alpha(\theta) = E[D|L = q_\alpha]$ is a conditional expectation that is conditioned on a probability zero region, $\{L = q_\alpha\}$, and we cannot directly simulate observations from this region.

To circumvent this difficulty, our kernel estimator assigns a weight to each of the observations depending on how close they are to the region, and it lets the weights go to zero asymptotically (i.e., as $n \to \infty$) if the observations are not in the region. To understand the rate of convergence of the kernel estimator, we consider the simplest kernel, $K(y) = 1_{\{-1/2<y<1/2\}}$, which is the density of a uniform distribution over the interval $[-1/2, 1/2]$. Then,

$$\bar{V}_n = \frac{\sum_{i=1}^n D_i \cdot 1_{\{-\delta_n/2<L_i-\hat{q}_\alpha^n<\delta_n/2\}}}{\sum_{i=1}^n 1_{\{-\delta_n/2\leq L_i-\hat{q}_\alpha^n\leq\delta_n/2\}}},$$

and it is the average of those D_i's such that $\hat{q}_\alpha^n - \delta_n/2 < L_i < \hat{q}_\alpha^n + \delta_n/2$. Intuitively, the kernel estimator approximates the region by $\{L \in (q_\alpha - \delta_n/2, q_\alpha + \delta_n/2)\}$ and approximates q_α by $\hat{q}_\alpha^n$ when it is unknown. Because $\Pr\{L \in (q_\alpha - \delta_n/2, q_\alpha + \delta_n/2)\} \approx f_L(q_\alpha)\delta_n$, the expected number of observations in the approximated region is $O(n\delta_n)$. This explains intuitively why the rate of convergence of $\bar{V}_n$ is $(n\delta_n)^{-1/2}$. When $n\delta_n^5 \to c$ with $c > 0$, the rate of convergence of $\bar{V}_n$ is $n^{-2/5}$ and this is the best that $\bar{V}_n$ may achieve.

Hong's [9] batching estimator uses another approach to approximate the region $\{L = q_\alpha\}$. It divides the n observations into k batches with each batch containing m observations. Then it approximates the region by $\{L = \hat{q}_\alpha^{m,j}\}$ for $j = 1, \ldots, k$, where $\hat{q}_\alpha^{m,j}$ is the $\lceil m\alpha \rceil$-th order statistic from the m observations in the j-th batch. Because there are k batches, the conditional expectation can be estimated and the rate of convergence is $k^{-1/2}$. Hong [9] further shows that the best rate of convergence of the batching estimator is $n^{-1/3}$ where $k/m^2 \to a$ with $a > 0$ as $n \to \infty$. Therefore, the kernel estimator has a faster rate of convergence than the batching estimator when both are set optimally.

Based on Theorem 2, we are able to construct confidence intervals for $q'_\alpha(\theta)$. When we set $n\delta_n^5 \to 0$, the asymptotic normal distribution has a zero mean. Then, we only need to estimate σ_∞^2 consistently. Let

$$\bar{G}_n(y) = \frac{1}{n\delta_n}\sum_{i=1}^n K\left(\frac{y - L_i}{\delta_n}\right) \cdot D_i^2.$$

Similar to the analysis of $\bar{V}_n(\hat{q}_\alpha^n)$, we can show that $\bar{G}_n(\hat{q}_\alpha^n)$ is a consistent estimator of $g_2(q_\alpha)$. Then, by its definition in Lemma 4, σ_∞^2 can be consistently estimated by

$$S_n^2 = \frac{\bar{G}_n(\hat{q}_\alpha^n)\bar{Q}_n(\hat{q}_\alpha^n) - \bar{R}_n^2(\hat{q}_\alpha^n)}{\bar{Q}_n^3(\hat{q}_\alpha^n)}\int_{-\infty}^{\infty} K^2(t)dt.$$

Therefore, an asymptotically valid $100(1-\beta)\%$ confidence interval of $q'_\alpha(\theta)$ is

$$\left(\bar{V}_n - z_{1-\beta/2}S_n/\sqrt{n\delta_n},\, \bar{V}_n + z_{1-\beta/2}S_n/\sqrt{n\delta_n}\right),$$

where $z_{1-\beta/2}$ is the $1 - \beta/2$ quantile of the standard normal distribution.

4. STEADY-STATE SIMULATION

In many examples, we are interested in the steady-state behaviors of a system. Then, we may use a steady-state simulation (Law and Kelton [23]). Suppose that $\{(L_i, D_i), i = 1, 2, \ldots, n\}$ are observations from a steady-state simulation. They are typically stationary and dependent. In this section, we show that the consistency and asymptotic normality still hold for $\bar{V}_n$ in the case of a steady-state simulation.

Suppose that $\{(L_i, D_i), i = 1, 2, \ldots\}$ is a stationary sequence. Let $\mathcal{F}_k$ be the σ-algebra generated by $\{(L_i, D_i), i = 1, 2, \ldots, k\}$ and $\mathcal{G}_k$ be the σ-algebra generated by $\{(L_i, D_i), i = k, k+1, \ldots\}$. Following Billingsley [26], let

$$\phi(k) = \sup\{|\Pr(B|A) - \Pr(B)| : A \in \mathcal{F}_s, \Pr(A) > 0, B \in \mathcal{G}_{s+k}\}.$$

Then, the sequence is ϕ-mixing if $\phi(k) \to 0$ as $k \to \infty$. The condition means that the dependence between time s and time $s + k$ goes to zero as k goes to infinity. Therefore, there does not exist a long-range dependence. Many stochastic processes are ϕ-mixing, including stationary Markov processes with a finite state space (Billingsley [27]) and positive recurrent regenerative processes (Glynn and Iglehart [28]). In this section, we make the following assumption on $\{(L_i, D_i), i = 1, 2, \ldots\}$:

ASSUMPTION 6: The sequence $\{(L_i, D_i), i = 1, 2, \ldots\}$ satisfies $\sum_{k=1}^{\infty} \sqrt{\phi(k)} < \infty$.

The same assumption is used or implied by many others. For instance, Heidelberger and Lewis [29] use it to analyze steady-state quantile estimators; Hong and Liu [11] use it to study steady-state probability sensitivity estimators; and Chien et al. [30] and Schruben [31] use assumptions that imply Assumption 6 to study the estimators of steady-state means. Note that the assumption implies that $\phi(k) \to 0$ as $k \to \infty$. Therefore, $\{(L_i, D_i), i = 1, 2, \ldots\}$ is a ϕ-mixing sequence when the assumption is satisfied.

4.1. Consistency

We consider the consistency of $\bar{V}_n$ for ϕ-mixing observations. The analysis is similar to that for the i.i.d. observations. Because the consistency of $\bar{R}_n(q_\alpha)$ and $\bar{Q}_n(q_\alpha)$ is known in the literature, we only need to prove that both $\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)$ and $\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)$ converge to 0 in probability. Then by Eq. (3) we conclude that $\bar{V}_n$ is consistent.

We summarize the consistency of $\bar{R}_n(q_\alpha)$ and $\bar{Q}_n(q_\alpha)$ in the following lemma. The result of the lemma is standard (see, for instance, Bosq [14] and Roussas [20]), but under different sets of conditions. We provide the proof in Liu and Hong [24] for completeness.

LEMMA 6: Suppose that Assumptions 4–6 are satisfied. If $\delta_n \to 0$ and $n\delta_n \to \infty$ as $n \to \infty$, then $\bar{R}_n(q_\alpha) \xrightarrow{P} g_1(q_\alpha)$ and $\bar{Q}_n(q_\alpha) \xrightarrow{P} g_0(q_\alpha)$.

Similar to Lemma 3 of the i.i.d. case, we have the following lemma for the dependent case. The proof of the lemma is provided in the Appendix.

LEMMA 7: Suppose that Assumptions 4–6 are satisfied. If $\delta_n \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} E\left[\frac{1}{n\delta_n^{m+1}} \sum_{i=1}^{n} K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right) \cdot D_i\right] = g_1^{(m)}(q_\alpha),$$

$$\limsup_{n\to\infty} n\delta_n^{2m+1} \mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}} \sum_{i=1}^{n} K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right) \cdot D_i\right] \le B,$$

for $m = 1, 2, 3, 4$, and some constant $B > 0$.

Similar to the discussion in the i.i.d. case, we can show that $\bar{V}_n$ is a consistent estimator of $q'_\alpha(\theta)$ using Lemmas 6 and 7. The proof is similar to that of Theorem 1 and thus is omitted here. To summarize, we have the following theorem on the consistency of $\bar{V}_n$ for ϕ-mixing observations.

THEOREM 3: Suppose that Assumptions 1–6 are satisfied, $E(D^2) < \infty$, $\delta_n \to 0$ as $n \to \infty$, and $\sup_n (n\delta_n^{5/2})^{-1} < \infty$. Then $\bar{V}_n$ is a consistent estimator of $q'_\alpha(\theta)$.

For steady-state simulations, Hong [9] conjectures that the batching estimator is also consistent and numerically verifies its consistency in a queueing example. With Theorem 3, we prove that our kernel estimator is indeed consistent.

4.2. Asymptotic Normality

Similar to the discussion in the i.i.d. case, to prove the asymptotic normality of $\bar{V}_n$ for dependent observations, the key is to show that $\sqrt{n\delta_n}[\bar{V}_n - \bar{V}_n(q_\alpha)] \xrightarrow{P} 0$ as $n \to \infty$.

Define

$$R_{n,i} = \frac{g_0(q_\alpha)}{\sqrt{n\delta_n}}(D_i \cdot K_{n,i} - E[D_i \cdot K_{n,i}]) - \frac{g_1(q_\alpha)}{\sqrt{n\delta_n}}(K_{n,i} - E[K_{n,i}]),$$

where $K_{n,i}$ denotes $K(\frac{q_\alpha - L_i}{\delta_n})$. Let $\rho_{n,i} = \mathrm{Cov}(R_{n,1}, R_{n,i+1})/\mathrm{Var}(R_{n,1})$ and $\beta_n = 1 + 2\sum_{i=1}^{n-1}(1 - i/n)\rho_{n,i}$. Then,

$\mathrm{Var}[\sum_{i=1}^{n} R_{n,i}] = n\mathrm{Var}[R_{n,1}]\beta_n$. By the covariance inequality for the ϕ-mixing sequence (Billingsley [27]), we have

$$\mathrm{Var}\left[\sum_{i=1}^{n} R_{n,i}\right] \le n\left[\mathrm{Var}[R_{n,1}] + 2\sum_{i=1}^{n-1} |\mathrm{Cov}(R_{n,1}, R_{n,i+1})|\right] \le n\mathrm{Var}[R_{n,1}]\left[1 + 4\sum_{i=1}^{n}\sqrt{\phi(i)}\right].$$

Then, we have

$$\beta_n = 1 + 2\sum_{i=1}^{n-1}(1 - i/n)\rho_{n,i} \le 1 + 4\sum_{i=1}^{n}\sqrt{\phi(i)}.$$

Because $1+4\sum_{i=1}^{n}\sqrt{\phi(i)}$ converges as $n\to\infty$ by Assumption 6, then, by the ratio comparison test (Marsden and Hoffman [32]), β_n converges as $n \to \infty$. We denote by β_∞ the limit of β_n, i.e., $\lim_{n\to\infty}\beta_n = \beta_\infty$.

Then, we have the following lemma on the asymptotic normality of $\bar{V}_n(q_\alpha)$. Similar results have been shown in Bosq [14] and Roussas and Tran [21] under different sets of conditions. For completeness, we provide our proof of the lemma in Liu and Hong [24].

LEMMA 8: Suppose that Assumptions 1–6 are satisfied, $E(|D|^{2+\gamma}) < \infty$ for some $\gamma > 0$ and $h_{2+\gamma}(t)$ is continuous at $t = q_\alpha$. If $\delta_n \to 0$ and $n\delta_n^5 \to c$ with $c \ge 0$ as $n \to \infty$, then

$$\sqrt{n\delta_n}\big[\bar{V}_n(q_\alpha) - q'_\alpha(\theta)\big] \Rightarrow \mu_\infty + \tilde{\sigma}_\infty \cdot N(0,1),$$

where $\tilde{\sigma}_\infty^2 = \beta_\infty \cdot \sigma_\infty^2$ and μ_∞ and σ_∞^2 are defined in Lemma 4.

To show the asymptotic normality of $\bar{V}_n$, we need the following lemma that is proved in the Appendix.

LEMMA 9: Suppose that Assumptions 4–6 are satisfied, $\delta_n \to 0$ as $n \to \infty$, and $\sup_n(n\delta_n^3)^{-1} < \infty$. Then, $\sqrt{n\delta_n}[\bar{V}_n - \bar{V}_n(q_\alpha)] \xrightarrow{P} 0$.

Combining Lemmas 8 and 9, we have the following theorem on the asymptotic normality of $\bar{V}_n$ in the case of steady-state simulations.

THEOREM 4: Suppose that Assumptions 1–6 are satisfied, $E(|D|^{2+\gamma}) < \infty$ for some $\gamma > 0$ and $h_{2+\gamma}(t)$ is continuous at $t = q_\alpha$. If $\delta_n \to 0$ and $n\delta_n^5 \to c$ with $c \ge 0$ as $n \to \infty$, and $\sup_n(n\delta_n^3)^{-1} < \infty$, then,

$$\sqrt{n\delta_n}\big[\bar{V}_n - q'_\alpha(\theta)\big] \Rightarrow \mu_\infty + \tilde{\sigma}_\infty \cdot N(0,1),$$

where μ_∞ and $\tilde{\sigma}_\infty$ are defined in Lemma 8.

Theorem 4 shows that the rate of convergence of $\bar{V}_n$ for steady-state simulations is the same as that for terminating simulations, which is $(n\delta_n)^{-1/2}$. When $n\delta_n^5 \to c$ with $c > 0$, the rate of convergence of $\bar{V}_n$ is $n^{-2/5}$ and this is the best that $\bar{V}_n$ may achieve.

Theorem 4 may also be used to construct confidence intervals for $q'_\alpha(\theta)$. To do so, we need to estimate the asymptotic variance $\tilde{\sigma}_\infty^2$. Because the observations $\{(L_i, D_i), i = 1, 2, \ldots\}$ are dependent, we suggest using the batch means method to estimate $\tilde{\sigma}_\infty^2$. We divide the n observations of (L_i, D_i) into k adjacent batches with each batch containing m observations. We require that both $m \to \infty$ and $k \to \infty$ as $n \to \infty$. For instance, a reasonable choice may be $m = k = \sqrt{n}$ if $\sqrt{n}$ is an integer. Let

$$\bar{V}_m^{(j)} = \frac{\sum_{i=1}^{m} K\left(\frac{\hat{q}_\alpha^{m,j} - L_{(j-1)m+i}}{\delta_m}\right)\cdot D_{(j-1)m+i}}{\sum_{i=1}^{m} K\left(\frac{\hat{q}_\alpha^{m,j} - L_{(j-1)m+i}}{\delta_m}\right)},$$

where $\hat{q}_\alpha^{m,j}$ denotes the α sample quantile obtained from the j-th batch of m observations. Then, the estimator of $\tilde{\sigma}_\infty^2$, denoted by $\tilde{S}_n^2$, can be expressed as

$$\tilde{S}_n^2 = \frac{m\delta_m}{k-1}\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - \frac{1}{k}\sum_{j=1}^{k}\bar{V}_m^{(j)}\right)^2. \tag{5}$$

Note that $\tilde{S}_n^2/m\delta_m$ is the sample variance of $\bar{V}_m^{(1)}, \ldots, \bar{V}_m^{(k)}$, which are approximately independent when k and m are large. Then, $\tilde{S}_n^2/m\delta_m$ is an estimator of $\mathrm{Var}(\bar{V}_m^{(1)})$. Because, by the central limit theorem, we know that $m\delta_m\mathrm{Var}(\bar{V}_m^{(1)}) \to \tilde{\sigma}_\infty^2$, $\tilde{S}_n^2$ is an estimator of $\tilde{\sigma}_\infty^2$. In the Appendix, we provide a rigorous proof to show that $\tilde{S}_n^2$ is indeed a consistent estimator of $\tilde{\sigma}_\infty^2$ under appropriate conditions. Then, an asymptotically valid $100(1-\beta)\%$ confidence interval of $q'_\alpha(\theta)$ is

$$\left(\bar{V}_n - z_{1-\beta/2}\tilde{S}_n/\sqrt{n\delta_n}, \quad \bar{V}_n + z_{1-\beta/2}\tilde{S}_n/\sqrt{n\delta_n}\right).$$

5. SELECTION OF δ_n

In the previous sections, we have shown that the kernel estimator we propose is consistent and asymptotically normally distributed for both i.i.d. and ϕ-mixing observations. To implement the estimator, however, we need to know how to select an appropriate δ_n. In this section, we propose a heuristic approach to solving this problem.

Typically, we are interested in selecting δ_n to minimize the asymptotic mean square error (MSE) of $\bar{V}_n$. Then, the key of this problem is finding the asymptotic variance and bias of $\bar{V}_n$. By Theorems 2 and 4, the asymptotic variance of $\bar{V}_n$ is $\sigma^2/(n\delta_n)$, where $\sigma^2 = \sigma_\infty^2$ for i.i.d. observations and

$\sigma^2 = \tilde{\sigma}_\infty^2$ for ϕ-mixing observations. Then, we only need to find the asymptotic bias of $\bar{V}_n$.

Firstly, we note that

$$\bar{V}_n - \bar{V}_n(q_\alpha) = \frac{1}{\bar{Q}_n(\hat{q}_\alpha^n)\bar{Q}_n(q_\alpha)}\big(\bar{Q}_n(q_\alpha)\big[\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)\big] - \bar{R}_n(q_\alpha)\big[\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)\big]\big).$$

By Taylor's expansion,

$$\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha) \approx (\hat{q}_\alpha^n - q_\alpha) \cdot \frac{1}{n\delta_n^2}\sum_{i=1}^n D_i \cdot K'\left(\frac{q_\alpha - L_i}{\delta_n}\right).$$

By Lemma 3, we know that $\frac{1}{n\delta_n^2}\sum_{i=1}^n D_i \cdot K'(\frac{q_\alpha - L_i}{\delta_n})$ converges to $g_1'(q_\alpha)$ in probability. Then, it is reasonable to expect that $E[\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)]$ is of the same order as $E(\hat{q}_\alpha^n) - q_\alpha$. It is known that, under some mild conditions, $E(\hat{q}_\alpha^n) - q_\alpha$ is of order $1/n$ (see, for instance, David [33]). Hence, $E[\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)]$ is of order $1/n$. A similar argument holds for $E[\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)]$ and hence we expect that $E[\bar{V}_n - \bar{V}_n(q_\alpha)]$ is of order $1/n$.

Furthermore, we note that

$$\bar{V}_n(q_\alpha) - q_\alpha'(\theta) = \frac{1}{g_0(q_\alpha)\bar{Q}_n(q_\alpha)}(g_0(q_\alpha)[\bar{R}_n(q_\alpha) - g_1(q_\alpha)] - g_1(q_\alpha)[\bar{Q}_n(q_\alpha) - g_0(q_\alpha)]).$$

Because $\bar{Q}_n(q_\alpha)$ converges to $g_0(q_\alpha)$ in probability, it is reasonable to approximate $E[\bar{V}_n(q_\alpha)] - q_\alpha'(\theta)$ by $\frac{1}{g_0^2(q_\alpha)}E(g_0(q_\alpha)[\bar{R}_n(q_\alpha) - g_1(q_\alpha)] - g_1(q_\alpha)[\bar{Q}_n(q_\alpha) - g_0(q_\alpha)])$, which equals $\mu_b\delta_n^2 + o(\delta_n^2)$ (see Eqs. (26) and (27) in Liu and Hong [24]) with

$$\mu_b = \frac{g_1''(q_\alpha) - q_\alpha'(\theta)g_0''(q_\alpha)}{g_0(q_\alpha)}\int_{-\infty}^{\infty} t^2K(t)dt.$$

In other words, $E[\bar{V}_n(q_\alpha)] - q_\alpha'(\theta) \approx \mu_b\delta_n^2 + o(\delta_n^2)$. Because $n\delta_n^2 \to \infty$ as $n \to \infty$, $E[\bar{V}_n - \bar{V}_n(q_\alpha)]$ is of a smaller order compared to $E[\bar{V}_n(q_\alpha)] - q_\alpha'(\theta)$, and hence can be neglected. Therefore, the bias of $\bar{V}_n$ is approximately $\mu_b\delta_n^2 + o(\delta_n^2)$.

The mean square error (MSE) is the summation of the variance and the squared bias. Then, ignoring small-order terms, we can approximate the asymptotic MSE of $\bar{V}_n$ by $\sigma^2/n\delta_n + \mu_b^2\delta_n^4$. To minimize the above asymptotic MSE, it is easy to see that the asymptotically optimal δ_n can be set as $\delta_n^* = d^* \cdot n^{-1/5}$, where $d^* = (\sigma^2/(4\mu_b^2))^{1/5}$.

With the above heuristic analysis, a pilot simulation can be run to estimate d^* during the implementation of our method. To do so, we need to estimate σ^2 and μ_b in the pilot simulation. The asymptotic variance, σ^2, can be consistently estimated by S_n^2 for i.i.d. observations and by $\tilde{S}_n^2$ for ϕ-mixing observations. We only need to consider how to estimate μ_b. We suggest estimating $g_0''(q_\alpha)$ and $g_1''(q_\alpha)$ by a finite difference method. Let s be the step size of the finite difference method. We may estimate $g_1''(q_\alpha)$ and $g_0''(q_\alpha)$ by

$$\hat{g}_1'' = \frac{\bar{R}_n(\hat{q}_\alpha^n + s) + \bar{R}_n(\hat{q}_\alpha^n - s) - 2\bar{R}_n(\hat{q}_\alpha^n)}{s^2},$$
$$\hat{g}_0'' = \frac{\bar{Q}_n(\hat{q}_\alpha^n + s) + \bar{Q}_n(\hat{q}_\alpha^n - s) - 2\bar{Q}_n(\hat{q}_\alpha^n)}{s^2}$$

respectively. Then, μ_b can be estimated by

$$\hat{\mu}_b = \frac{\hat{g}_1'' - \bar{V}_n\hat{g}_0''}{\bar{Q}_n(\hat{q}_\alpha^n)}\int_{-\infty}^{\infty} t^2K(t)dt.$$

In the pilot simulation, we suggest setting $\delta_n = n^{-1/5}$ initially and estimating d^* by $\hat{d} = (S_n^2/(4\hat{\mu}_b^2))^{1/5}$ for i.i.d. observations and $\hat{d} = (\tilde{S}_n^2/(4\hat{\mu}_b^2))^{1/5}$ for ϕ-mixing observations. Then, we may set the new δ_n to $\hat{d} \cdot n^{-1/5}$ and estimate d^* again. After several iterations, we stop and obtain an estimate of d^*, denoted as $\hat{d}^*$. We may use $\delta_n = \hat{d}^* \cdot n^{-1/5}$ in the main simulation.

To construct confidence intervals for $q_\alpha'(\theta)$, we require that the limiting distributions in Theorems 2 and 4 be normally distributed with zero means; in other words, $n\delta_n^5 \to 0$ as $n \to \infty$. However, $d^* \cdot n^{-1/5}$ violates this requirement. To fix this, we suggest letting $\delta_n = d^* \cdot n^{-1/3}$ when constructing confidence intervals. Intuitively, this choice of δ_n guarantees that $\bar{V}_n$ has a smaller bias and a larger variance so that the confidence interval is asymptotically valid.

6. NUMERICAL EXAMPLES

In this section, we show the performance of the kernel estimator using three examples, including a portfolio management example, a production-inventory example, and a queueing example. These examples are borrowed from Hong [9] for fair comparison of our kernel estimator and Hong's [9] batching estimator. During the implementation, the kernel K is chosen to be the standard normal density function, and δ_n is selected by the selection procedure in Section 5. Numerical results reported are based on 1000 independent replications.

6.1. A Portfolio Management Example

A portfolio is composed of three assets. The annual rates of return of the assets are denoted as X_1, X_2, and X_3, respectively, and the percentages of the total fund allocated to the assets are denoted as θ_1, θ_2, and θ_3, respectively. Suppose that $X = (X_1, X_2, X_3)'$ follows a multivariate

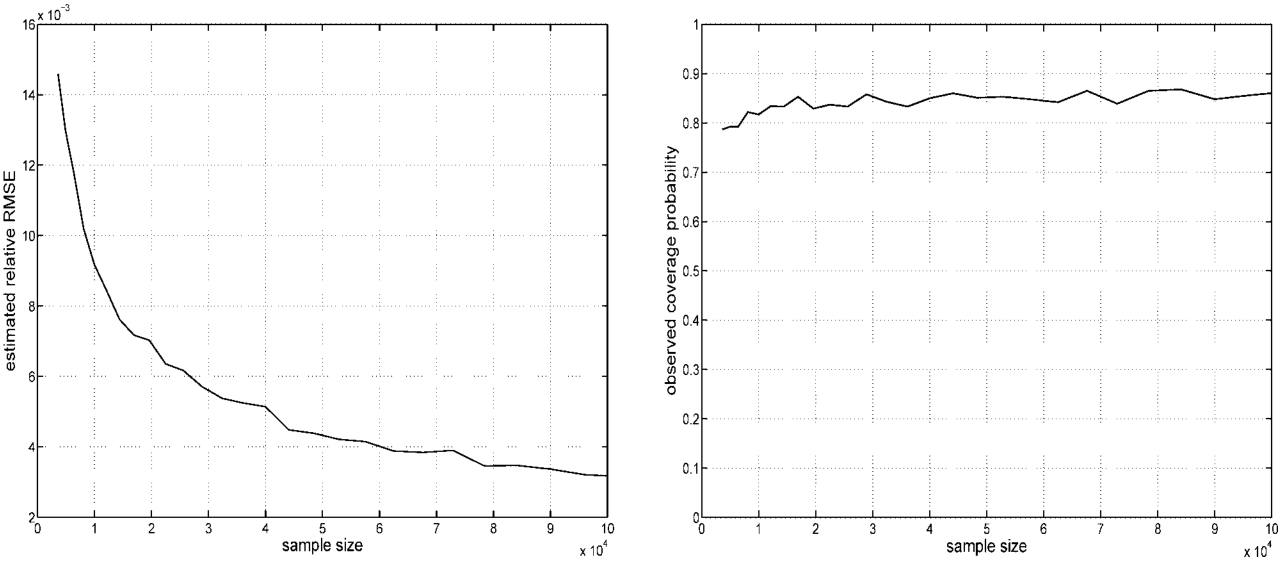

Figure 1. The performance of $\bar{V}_n$ and its 90% confidence interval in the portfolio management example.

normal distribution with a mean $\mu = (0.06, 0.15, 0.25)'$ and a variance-covariance matrix

$$\Sigma = \begin{pmatrix} 0.02 & & \\ & 0.10 & \\ & & 0.22 \end{pmatrix} \begin{pmatrix} 1 & -0.3 & -0.2 \\ -0.3 & 1 & 0.2 \\ -0.2 & 0.2 & 1 \end{pmatrix} \times \begin{pmatrix} 0.02 & & \\ & 0.10 & \\ & & 0.22 \end{pmatrix}.$$

Then, the annual rate of return of the portfolio is

$$L(\theta) = \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3,$$

which follows a normal distribution with mean $\theta'\mu$ and variance $\theta'\Sigma\theta$. Then, the quantile and quantile sensitivity of $L(\theta)$ can be calculated analytically. Suppose that we are interested in the quantile sensitivity with respect to θ_3, with $\theta = (0.2, 0.3, 0.5)'$. Then, $\partial q_\alpha(\theta)/\partial\theta_3 = 0.25 + 0.2135 z_\alpha$. We use the kernel estimator to estimate the quantile sensitivity and compare it to the theoretical value. We also compare its performance to Hong's [9] batching estimator.

Note that the pathwise derivative is $D(\theta) = \partial L(\theta)/\partial\theta_3 = X_3$. Then, the proposed kernel estimator can be easily applied. To test its performance, we let $\alpha = 0.9$ and plot the estimated relative RMSE (square root of MSE), defined as $\sqrt{\text{MSE}(\bar{V}_n)}/|q'_\alpha(\theta)|$, in the left panel of Fig. 1. We see that the estimation error decreases as the sample size increases. When the sample size is 10,000, the relative error is smaller than 1%. We also plot the observed coverage probabilities of the 90% confidence interval in the right panel of Fig. 1. The plot shows that the coverage probabilities are close to 90%, which is the nominal coverage probability. To summarize, the plots in Fig. 1 coincide with the theoretical asymptotic properties that are analyzed in Section 3.

To compare the kernel estimator with Hong's [9] batching estimator, we plot the MSE ratio of the batching estimator to the kernel estimator, as well as the half-length ratio of their confidence intervals, in the left and right panels of Fig. 2, respectively. The results show that the kernel estimator has a smaller MSE, and its confidence intervals have smaller half lengths, which means the estimation is more accurate, than the batching estimator. Furthermore, when the sample size becomes larger, the kernel estimator becomes more preferable than the batching estimator.

6.2. A Production-Inventory Example

A capacitated production system operates under a base-stock inventory policy. It has a base stock level $s > 0$, and it has a capacity for producing at maximum c units per period. Within each period, the products from the last period first arrive. Then, the demands of the period occur, and they are filled or backlogged based on the available inventory. At the end of the period, the production amount is determined. Let I_i be the inventory minus the backlog in period i, A_i, and R_i be the demand and production amount in period i. Then, the system evolves as follows (Glasserman and Tayur [34]): $I_{i+1} = I_i - A_i + R_{i-1}$ and $R_i = \min\{c, [s + A_i - (I_i + R_{i-1})]^+\}$, where $a^+ = \max\{a, 0\}$.

We further assume that there are linear holding and backorder costs. The holding cost is h per unit per period and the backorder cost is b per unit per period. Let c_i be the cost of period i. Then, $c_i = h(R_{i-1} + I_i^+) + bI_i^-$, where $a^- = -\min\{a, 0\}$. Let $L(s, A) = \sum_{i=1}^{n} c_i$ be the total cost over the first n periods where $A = (A_1, A_2, \ldots, A_n)$.

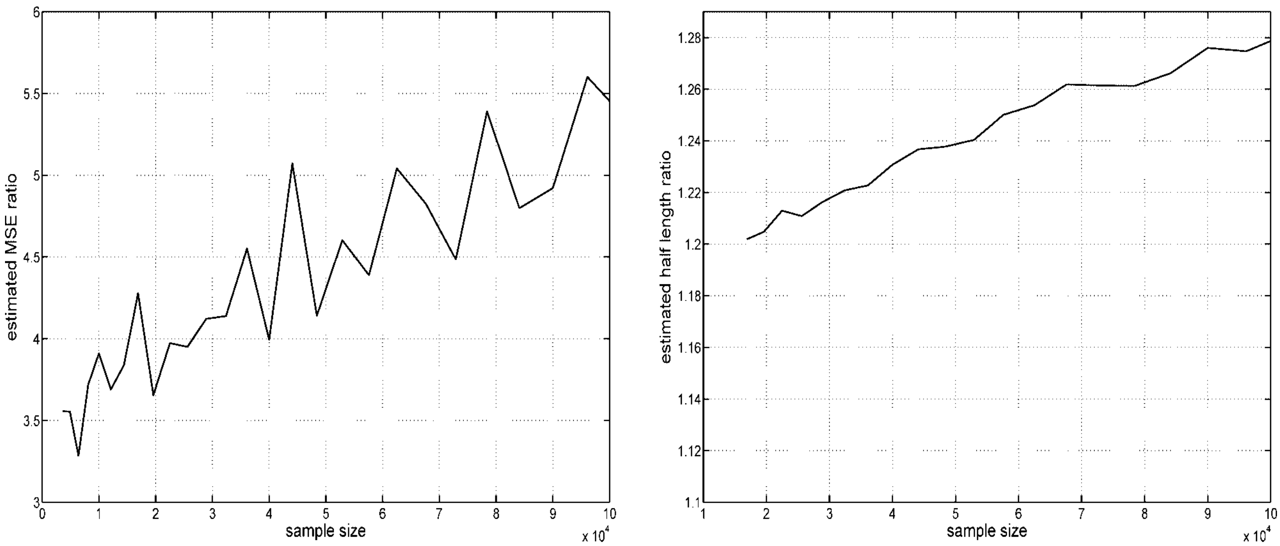

Figure 2. A comparison of the performance of $\bar{V}_n$ with the performance of the batching estimator in the portfolio management example.

Glasserman and Tayur [34] have studied this problem under a more general setting where they are interested in the sensitivity of the expected total cost. However, when the decision maker is risk averse, he/she may be interested in the α quantile of the total cost for some $\alpha \geq 0.5$, rather than the expected total cost. In this case, it is important to know the quantile sensitivity of $L(s, A)$ with respect to the base stock level s, which is our goal.

We assume s to be a continuous decision variable and set $s = 1.5$, $c = 0.5$, $h = 0.1$, $b = 0.2$, $n = 20$, $I_1 = s$, and $R_0 = 0$, and let all A_i follow independent exponential distributions with rate 1 for all n periods. To compute $\partial L(s, A)/\partial s$, we need to know how to calculate $\partial c_i/\partial s$. Hong [9] shows that $\partial c_i/\partial s = 1_{\{I_i>0\}}h - 1_{\{I_i<0\}}b$. Then, $\partial L(s, A)/\partial s$ can be obtained and the kernel estimator can be easily applied.

An accurate estimate of the quantile sensitivity can be obtained by using Hong's [9] batching estimator with a very large sample size. Then, this estimate is used as a benchmark to test the performance of the estimators. We plot the relative RMSE of the kernel estimator and the observed coverage probabilities of its 90% confidence interval in the left and right panels of Fig. 3, respectively. We see that the relative error is less than 0.6% when the sample size is larger than 1000 and the confidence interval has the desired coverage probability, which conforms the theoretical asymptotic properties that are analyzed in Section 3.

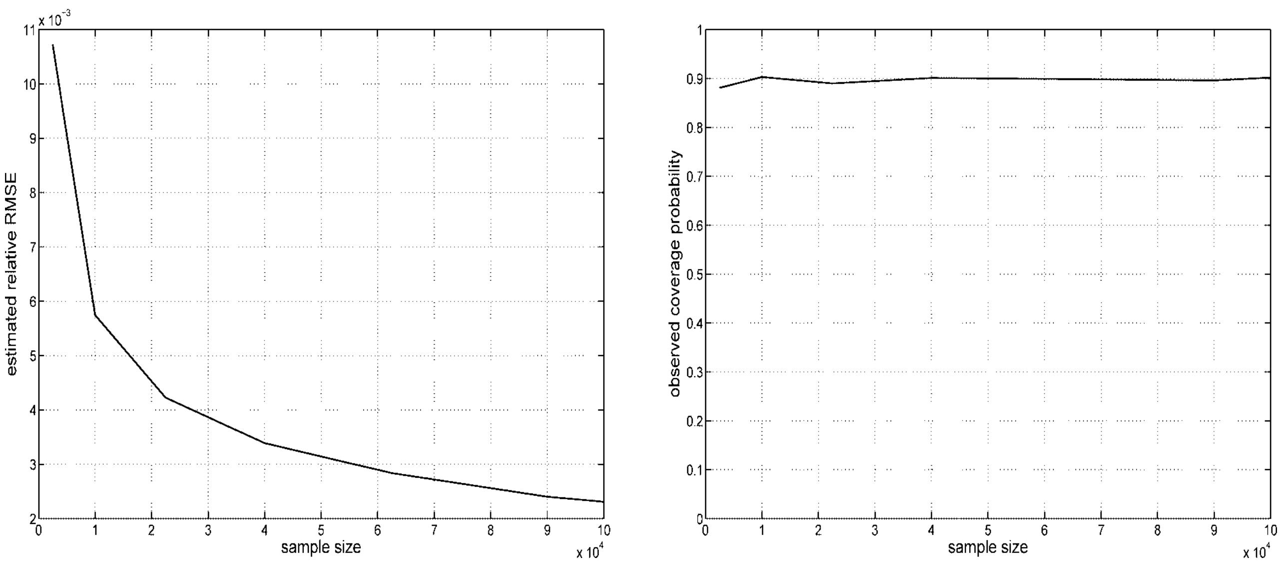

Figure 3. The performance of $\bar{V}_n$ and its 90% confidence interval in the production-inventory example.

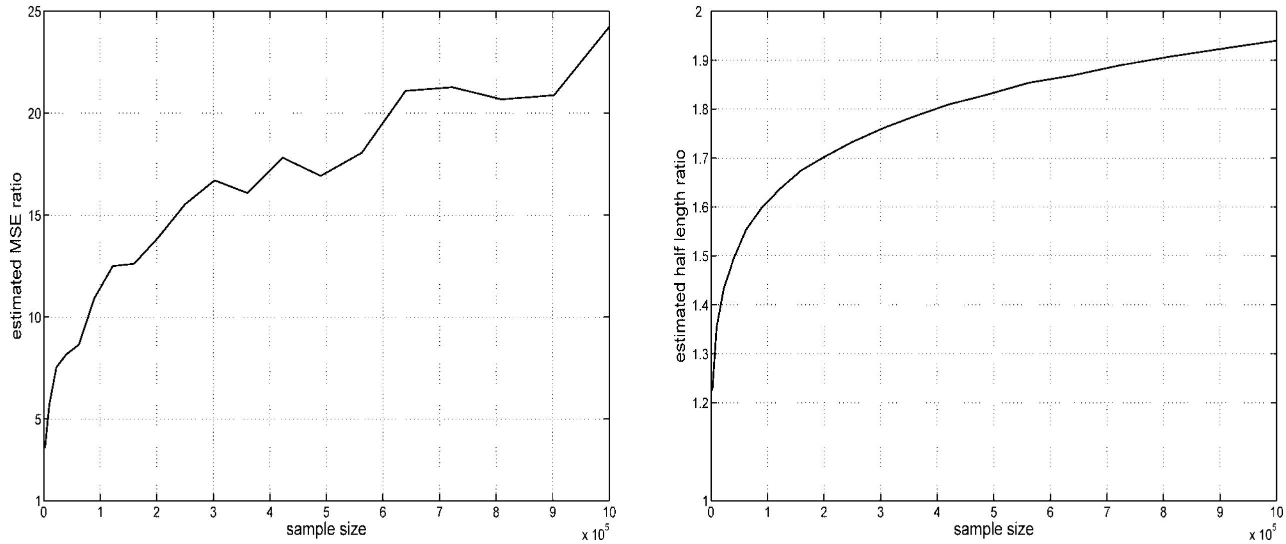

Figure 4. A comparison of the performance of $\bar{V}_n$ with the performance of the batching estimator in the production-inventory example.

To compare the performance of the kernel estimator and the batching estimator, we plot the MSE ratios of the batching estimator to the kernel one, as well as the half-length ratios of their confidence intervals, in the left and right panels of Fig. 4, respectively. The plots show that the kernel estimator always outperforms the batching estimator and becomes more preferable than the batching estimator when the sample size is large.

6.3. A Queueing Example

We use a queueing example to illustrate how the kernel estimator can be applied to steady-state simulations. Specifically, an $M/M/1$ queue is considered, where the observations are dependent. Let $\theta = (\theta_1, \theta_2)'$, where θ_1 and θ_2 denote the mean interarrival time and the mean service time, respectively. Let $L(\theta)$ be the customer's steady-state sojourn time. We are interested in estimating $\partial q_\alpha(\theta)/\partial\theta_2$, where $q_\alpha(\theta)$ is the α-quantile of $L(\theta)$. Then $\partial L(\theta)/\partial\theta_2$ can be obtained using the IPA method (Glasserman [13]).

When the queue is stable, that is, $\theta_1 > \theta_2$, $L(\theta)$ is exponentially distributed with a rate $1/\theta_2 - 1/\theta_1$ (Ross [35]). Therefore, for any $0 < \alpha < 1$,

$$\frac{\partial q_\alpha(\theta)}{\partial\theta_2} = -\left[\frac{\theta_1}{\theta_1 - \theta_2}\right]^2 \log(1 - \alpha).$$

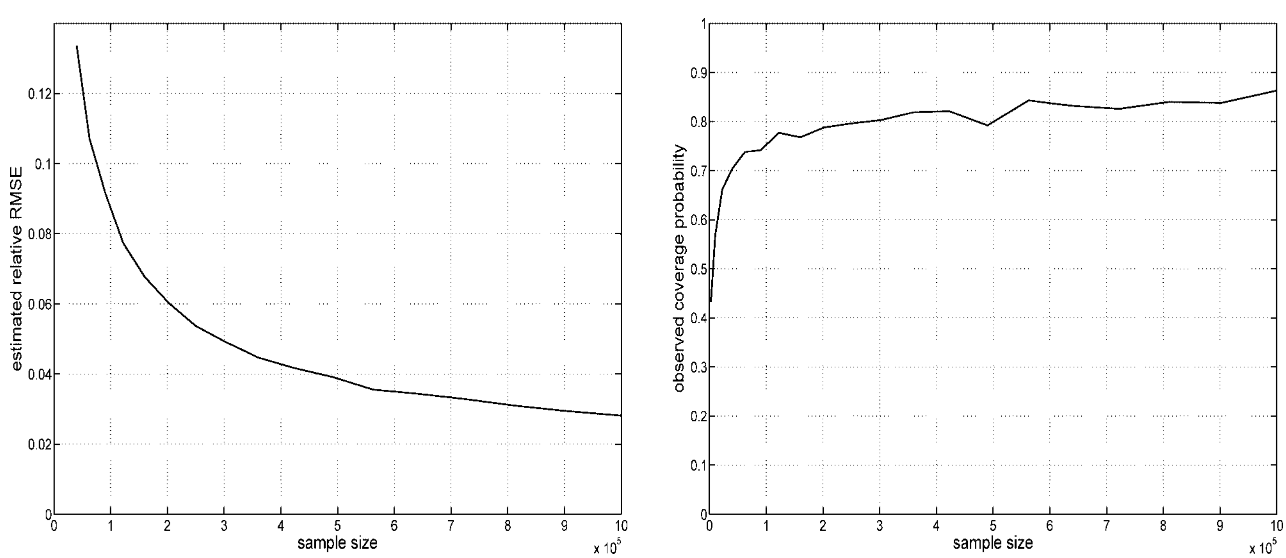

Figure 5. The performances of $\bar{V}_n$ and its 90% confidence interval in the queueing example.

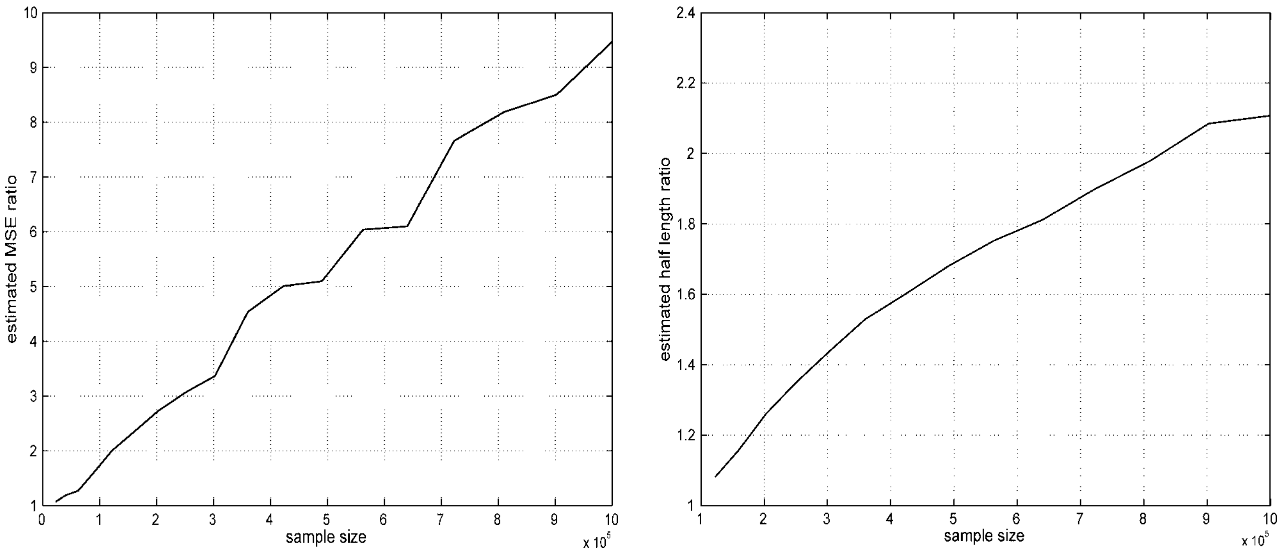

Figure 6. A comparison of the performance of $\bar{V}_n$ with the performance of the batching estimator in the queueing example.

Then, the actual value of $\partial q_\alpha(\theta)/\partial\theta_2$ can be computed. This enables us to test the performance of the kernel estimator.

We let $\theta_1 = 10, \theta_2 = 8$, and $\alpha = 0.9$, and plot the estimated relative RMSEs and the observed coverage probabilities of its 90% confidence interval in the left and right panels of Fig. 5, respectively. We see that relative error is smaller than 3% when the sample size is 90,000 and the observed coverage probabilities are close to 90% when the sample size is large. These results coincide with the asymptotic properties that are analyzed in Section 4.

Note that Hong [9] proves the asymptotic properties of the batching estimator only for i.i.d. observations. However, he numerically shows that the batching estimator also works for the $M/M/1$ example. To compare these estimators, we plot the MSE ratio of the batching estimator to the kernel estimator, as well as the half-length ratio of their confidence intervals, in the left and right panels of Fig. 6, respectively. We see that the kernel estimator outperforms the batching estimator in terms of both MSE and half length and becomes more preferable than the batching estimator when the sample size is large.

7. CONCLUSIONS

In this article, we propose a kernel estimator for estimating quantile sensitivities using stochastic simulation. We show the consistency and asymptotic normality of the estimator for both terminating and steady-state simulations. Numerical examples show that the estimator works well in practical problems, and it is more efficient than Hong's [9] batching estimator.

APPENDIX

Bochner's Lemma and A Direct Implication

In the later proofs, we need the following lemma that is known as Bochner's Lemma (see, e.g., Parzen [36]).

LEMMA 10 (Bochner's Lemma): Suppose that $H(y)$ is a function such that $\sup_{-\infty<y<\infty} |H(y)| < \infty$, $\int_{-\infty}^{+\infty} |H(y)|dy < \infty$ and $\lim_{y\to\infty} |yH(y)| = 0$. Let $f(y)$ satisfy $\int_{-\infty}^{+\infty} |f(y)|dy < \infty$ and let $\{b_n\}$ be a sequence of positive constants satisfying $\lim_{n\to\infty} b_n = 0$. Let

$$f_n(x) = \frac{1}{b_n}\int_{-\infty}^{+\infty} H\left(\frac{y}{b_n}\right) f(x-y)dy.$$

Then, at every point x of continuity of $f(\cdot)$,

$$\lim_{n\to\infty} f_n(x) = f(x)\int_{-\infty}^{+\infty} H(y)dy.$$

A direct implication of the Bochner's Lemma is stated as follows, which will be used repeatedly.

Suppose that $E[|D|^m] < \infty$ for some nonnegative number m. Then for any positive l,

$$\lim_{n\to\infty} E\left[\frac{1}{\delta_n}K^l\left(\frac{t-L}{\delta_n}\right)\cdot D^m\right] = g_m(t)\int_{-\infty}^{\infty} K^l(u)du \tag{6}$$

if $g_m(x)$ is continuous at $x = t$, and

$$\lim_{n\to\infty} E\left[\frac{1}{\delta_n}K^l\left(\frac{t-L}{\delta_n}\right)\cdot |D|^m\right] = h_m(t)\int_{-\infty}^{\infty} K^l(u)du \tag{7}$$

if $h_m(x)$ is continuous at $x = t$.

PROOF OF LEMMA 3: Let $K_{n,i}^{(m)}$ denote $K^{(m)}(\frac{q_\alpha - L_i}{\delta_n})$, for $m = 1,2,3,4$. We note that $E[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i] = \frac{1}{\delta_n^{m+1}}\int_{-\infty}^{\infty} K^{(m)}(\frac{q_\alpha - t}{\delta_n})g_1(t)dt$. Then, integrating by parts yields

$$\begin{aligned}\frac{1}{\delta_n^{m+1}}\int_{-\infty}^{\infty} K^{(m)}\left(\frac{q_\alpha - t}{\delta_n}\right)g_1(t)dt \\ = -\frac{1}{\delta_n^m}K^{(m)}\left(\frac{q_\alpha - t}{\delta_n}\right)g_1(t)\Big|_{-\infty}^{\infty} + \frac{1}{\delta_n^m}\int_{-\infty}^{\infty}K^{(m-1)}\left(\frac{q_\alpha - t}{\delta_n}\right)g_1^{(1)}(t)dt \\ = \frac{1}{\delta_n^m}\int_{-\infty}^{\infty}K^{(m-1)}\left(\frac{q_\alpha - t}{\delta_n}\right)g_1^{(1)}(t)dt.\end{aligned}$$

Similarly, we can show that $\frac{1}{\delta_n^m}\int_{-\infty}^{\infty}K^{(m-1)}(\frac{q_\alpha - t}{\delta_n})g_1^{(1)}(t)dt = \frac{1}{\delta_n^{m-1}}\int_{-\infty}^{\infty}K^{(m-2)}(\frac{q_\alpha - t}{\delta_n})g_1^{(2)}(t)dt$. Then, iteratively we have $E[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i] = \frac{1}{\delta_n}\int_{-\infty}^{\infty}K(\frac{q_\alpha - t}{\delta_n})g_1^{(m)}(t)dt$.

Because $g_1^{(m)}(t)$ is continuous at $t = q_\alpha$, $\delta_n \to \infty$ as $n \to \infty$, and $\int_{-\infty}^{\infty}|g_1^{(m)}(t)|dt < \infty$, then by Bochner's Lemma,

$$\lim_{n\to\infty} E\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] = \lim_{n\to\infty} E\left[\frac{1}{\delta_n^{m+1}}K_{n,1}^{(m)} \cdot D_1\right] = g_1^{(m)}(q_\alpha).$$

Furthermore, note that

$$\begin{aligned}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] \\ = \frac{1}{n\delta_n^{2m+1}}E\left[\frac{1}{\delta_n}\left(K_{n,1}^{(m)}\right)^2 \cdot D_1^2\right] - \frac{1}{n}\left(E\left[\frac{1}{\delta_n^{m+1}}K_{n,1}^{(m)} \cdot D_1\right]\right)^2.\end{aligned}$$

Because $g_2(t)$ is continuous at $t = q_\alpha$ by Assumption 5, Assumption 4 holds, and $E(D^2) < \infty$. Then, similar to Eq. (6),

$$\lim_{n\to\infty} E\left[\frac{1}{\delta_n}\left(K_{n,1}^{(m)}\right)^2 \cdot D_1^2\right] = \sigma_m^2.$$

Therefore, $\lim_{n\to\infty} n\delta_n^{2m+1}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] = \sigma_m^2$. This concludes the proof. □

PROOF OF LEMMA 5: Note that

$$\begin{aligned}\sqrt{n\delta_n}(\bar{V}_n - \bar{V}_n(q_\alpha)) = \frac{\sqrt{n\delta_n}}{\bar{Q}_n(\hat{q}_\alpha^n)\bar{Q}_n(q_\alpha)} \\ \times \left[\bar{Q}_n(q_\alpha)\left(\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)\right) - \bar{R}_n(q_\alpha)\left(\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)\right)\right].\end{aligned}$$

In Lemma 2 and Theorem 1, we have shown that $\bar{Q}_n(q_\alpha) \xrightarrow{P} g_0(q_\alpha)$, $\bar{R}_n(q_\alpha) \xrightarrow{P} g_1(q_\alpha)$ and $\bar{Q}_n(\hat{q}_\alpha^n) \xrightarrow{P} g_0(q_\alpha)$. Then, it suffices to show that $\sqrt{n\delta_n}(\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha))$ and $\sqrt{n\delta_n}(\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha))$ converge to 0 in probability.

Let $K_{n,i}^{(m)}$ denote $K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right)$ for $m = 1,2,3,4$. Similar to Eq. (4),

$$\begin{aligned}\sqrt{n\delta_n}\left[\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)\right] \\ = \sqrt{n\delta_n}\frac{1}{n\delta_n}\sum_{i=1}^n\left[K\left(\frac{\hat{q}_\alpha^n - L_i}{\delta_n}\right) - K\left(\frac{q_\alpha - L_i}{\delta_n}\right)\right]D_i \\ = \sum_{m=1}^4\frac{1}{m!}\left[\sqrt{n}\left(\hat{q}_\alpha^n - q_\alpha\right)\right]^m\frac{\sqrt{n\delta_n}}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i \\ + o_p\left(\left[\sqrt{n}\left(\hat{q}_\alpha^n - q_\alpha\right)\right]^4\left(n\delta_n^2\right)^{-3/2}\frac{1}{n}\sum_{i=1}^n|D_i|\right). \quad (8)\end{aligned}$$

For $m = 1,2,3,4$, by Lemma 3,

$$\begin{aligned}\lim_{n\to\infty} E\left[\frac{\sqrt{n\delta_n}}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] \\ = \lim_{n\to\infty}\frac{\sqrt{n\delta_n}}{n^{m/2}}E\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] = 0,\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\mathrm{Var}\left[\frac{\sqrt{n\delta_n}}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] \\ = \lim_{n\to\infty}\frac{1}{\left(n\delta_n^2\right)^m} \cdot n\delta_n^{2m+1}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] = 0,\end{aligned}$$

where the last equation holds since $n\delta_n^2 \to \infty$ as $n \to \infty$ when $\sup_n(n\delta_n^3)^{-1} < \infty$.

Then, by Chebyshev's equality (Durrett [25]), for $m = 1,2,3,4$,

$$\frac{\sqrt{n\delta_n}}{n^{m/2}}\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i \xrightarrow{P} 0.$$

Recalling that $[\sqrt{n}(\hat{q}_\alpha^n - q_\alpha)]^m$ converges in distribution to some random variable, we know that the first term on the right-hand side (RHS) of Eq. (8) converges to 0 in probability. Moreover, $\frac{1}{n}\sum_{i=1}^n|D_i| \xrightarrow{P} E(|D|)$ by the strong law of large numbers. Then, the second term on the RHS of Eq. (8) converges to 0 in probability, since $\sup_n(n\delta_n^3)^{-1} < \infty$. Therefore, $\sqrt{n\delta_n}[\bar{R}_n(\hat{q}_\alpha^n) - \bar{R}_n(q_\alpha)] \xrightarrow{P} 0$. Similarly, we can show that $\sqrt{n\delta_n}[\bar{Q}_n(\hat{q}_\alpha^n) - \bar{Q}_n(q_\alpha)] \xrightarrow{P} 0$. This concludes the proof. □

PROOF OF LEMMA 7: The first half of the lemma is the same as that in Lemma 3. Hence, we only focus on the second half. Let $K_{n,i}^{(m)}$ denote $K^{(m)}\left(\frac{q_\alpha - L_i}{\delta_n}\right)$. Then, it suffices to show that

$$\limsup_{n\to\infty} n\delta_n^{2m+1}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] \le B.$$

Note that

$$\begin{aligned}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^n K_{n,i}^{(m)} \cdot D_i\right] = \frac{1}{n\delta_n^{2m+2}} \\ \times\left[2\sum_{i=1}^{n-1}\left(1 - \frac{i}{n}\right)\mathrm{Cov}\left(K_{n,1}^{(m)} \cdot D_1, K_{n,i+1}^{(m)} \cdot D_{i+1}\right) + \mathrm{Var}\left(K_{n,1}^{(m)} \cdot D_1\right)\right] \\ \le \frac{1}{n\delta_n^{2m+2}}\left[2\sum_{i=1}^{n-1}\left|\mathrm{Cov}\left(K_{n,1}^{(m)} \cdot D_1, K_{n,i+1}^{(m)} \cdot D_{i+1}\right)\right| + \mathrm{Var}\left(K_{n,1}^{(m)} \cdot D_1\right)\right] \\ \le \frac{1}{n\delta_n^{2m+2}}E\left[\left(K_{n,1}^{(m)} \cdot D_1\right)^2\right]\left[1 + 4\sum_{i=1}^{\infty}\sqrt{\phi(i)}\right],\end{aligned}$$

where the last inequality follows from the covariance inequality (Billingsley [27]).

By Bochner's Lemma, we know that

$$\lim_{n\to\infty}\frac{1}{\delta_n}E\left[\left(K_{n,1}^{(m)} \cdot D_1\right)^2\right] = \sigma_m^2,$$

where σ_m^2 is the same as the one defined in Lemma 3. Let $B = [1 + 4\sum_{i=1}^{\infty}\sqrt{\phi(i)}] \cdot \sigma_m^2$. Then,

$$\limsup_{n\to\infty} n\delta_n^{2m+1}\mathrm{Var}\left[\frac{1}{n\delta_n^{m+1}}\sum_{i=1}^{n}K_{n,i}^{(m)}\cdot D_i\right] \le B.$$

□

PROOF OF LEMMA 9: Sen [37] shows that $\sqrt{n}(\hat{q}_\alpha^n - q_\alpha)$ converges to a normal distribution when Assumption 6 is satisfied. Then, similar to the proof of Lemma 5, we can prove this lemma by using Lemma 7. □

Consistency of $\tilde{S}_n^2$

We show that under certain conditions, the variance estimator $\tilde{S}_n^2$ defined in Eq. (5) is consistent when $\{(L_1, D_1), \ldots, (L_n, D_n)\}$ satisfies Assumption 6. Let $\bar{A}_n = \frac{1}{k}\sum_{j=1}^{k}\bar{V}_m^{(j)}$. Then, we have the following result on the consistency of $\tilde{S}_n^2$.

LEMMA 11: Suppose that Assumptions 4–6 are satisfied, the sequences $\{k^2(\bar{A}_n - E[\bar{A}_n])^4, n \ge 1\ldots\}$ and $\{m^2\delta_n^2(\bar{V}_m^{(1)} - E[\bar{V}_m^{(1)}])^4, n \ge 1\ldots\}$ are uniformly integrable, $\delta_n \to 0$ as $n \to \infty$, $\limsup_n (n\delta_n^3)^{-1} < \infty$ and $\limsup_n m\delta_m/k < \infty$. Then, $\tilde{S}_n^2$ converges to $\tilde{\sigma}_\infty^2$ in probability as $n \to \infty$.

PROOF: Because $E[\bar{A}_n] = E[\bar{V}_m^{(1)}]$, by Eq. (5), we have

$$\tilde{S}_n^2 = \frac{m\delta_m}{k-1}\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - \bar{A}_n\right)^2 = \frac{m\delta_m}{k-1}\left[\sum_{j=1}^{k}\left(\bar{V}_m^{(j)}\right)^2 - k\bar{A}_n^2\right]$$
$$= \frac{m\delta_m}{k-1}\left\{\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - E\left[\bar{V}_m^{(1)}\right]\right)^2 - k(\bar{A}_n - E[\bar{A}_n])^2\right\}.$$

Because $m \to \infty$ as $n \to \infty$, by Theorem 4, $\sqrt{m\delta_m}(\bar{V}_m^{(1)} - E[\bar{V}_m^{(1)}]) \Rightarrow \tilde{\sigma}_\infty \cdot N(0, 1)$. Then, we have

$$m\delta_m E\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right] \to \tilde{\sigma}_\infty^2 \quad \text{and}$$
$$m^2\delta_m^2 E\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^4\right] \to 3\tilde{\sigma}_\infty^4, \quad (9)$$

because $\{m^2\delta_n^2(\bar{V}_m^{(1)} - E[\bar{V}_m^{(1)}])^4, n \ge 1\}$ is uniformly integrable.

Note that $\mathrm{Var}(\bar{A}_n) = \frac{1}{k^2}\mathrm{Var}(\sum_{j=1}^{k}\bar{V}_m^{(j)})$. Then,

$$\mathrm{Var}(\bar{A}_n) = \frac{1}{k^2}\Bigg[k\mathrm{Var}\left(\bar{V}_m^{(1)}\right) + 2\sum_{j=1}^{k-1}(k-j) \times \mathrm{Cov}\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right], \bar{V}_m^{(j+1)} - E\left[\bar{V}_m^{(1)}\right]\right)\Bigg]$$
$$\le \frac{1}{k^2}\left[k\mathrm{Var}\left(\bar{V}_m^{(1)}\right) + 2k\sum_{j=1}^{k-1}\left|\mathrm{Cov}\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right], \bar{V}_m^{(j+1)} - E\left[\bar{V}_m^{(1)}\right]\right)\right|\right]$$
$$\le \frac{1}{k^2}\left[k\mathrm{Var}\left(\bar{V}_m^{(1)}\right) + 4k\sum_{j=1}^{k-1}E\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right]\sqrt{\tilde{\phi}(j)}\right] \quad (10)$$

$$\le \frac{1}{k}E\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right]\left[1 + 4\sum_{j=1}^{k-1}\sqrt{\tilde{\phi}(j)}\right], \quad (11)$$

where Eq. (10) follows from the covariance inequality (Billingsley [27], lemma 1, page 170), and $\{\tilde{\phi}(j), j \ge 1\}$ are mixing coefficients of $\{\bar{V}_m^{(1)}, \ldots, \bar{V}_m^{(k)}\}$.

Note that $\tilde{\phi}(j) \le \phi(j)$ and $m\delta_m \to \infty$ as $n \to \infty$. Then, by Assumption 6 and Eqs. (9) and (11), we know that $k\mathrm{Var}(\bar{A}_n) \to 0$. Then, $\sqrt{k}(\bar{A}_n - E[\bar{A}_n]) \xrightarrow{P} 0$, and thus by the continuous mapping theorem (Durrett [25]), $k^2(\bar{A}_n - E[\bar{A}_n])^4 \xrightarrow{P} 0$. Hence,

$$k^2\mathrm{Var}[(\bar{A}_n - E[\bar{A}_n])^2] \to 0, \quad (12)$$

because $\{k^2(\bar{A}_n - E[\bar{A}_n])^4, n \ge 1\}$ is uniformly integrable.

Therefore,

$$E\left(\tilde{S}_n^2\right) = \frac{k}{k-1}m\delta_m E\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right] - \frac{k}{k-1}m\delta_m\mathrm{Var}(\bar{A}_n) \to \tilde{\sigma}_\infty^2, \quad (13)$$

because $m\delta_m E[(\bar{V}_m^{(1)} - E[\bar{V}_m^{(1)}])^2] \to \tilde{\sigma}_\infty^2$ by Eq. (9), $\limsup_n m\delta_m/k < \infty$ and $k\mathrm{Var}(\bar{A}_n) \to 0$ as $n \to \infty$.

Note that $\mathrm{Var}(X - Y) \le 2[\mathrm{Var}(X) + \mathrm{Var}(Y)]$. Then,

$$\mathrm{Var}\left(\tilde{S}_n^2\right) = \frac{m^2\delta_m^2}{(k-1)^2}\mathrm{Var}\left\{\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - E\left[\bar{V}_m^{(1)}\right]\right)^2 - k(\bar{A}_n - E[\bar{A}_n])^2\right\}$$
$$\le \frac{2m^2\delta_m^2}{(k-1)^2}\left\{\mathrm{Var}\left[\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right] + k^2\mathrm{Var}\left[\left(\bar{A}_n - E\left[\bar{A}_n\right]\right)^2\right]\right\}. \quad (14)$$

Similar to the proof of Eq. (11), we can show that

$$\mathrm{Var}\left[\sum_{j=1}^{k}\left(\bar{V}_m^{(j)} - E\left[\bar{V}_m^{(1)}\right]\right)^2\right] \le kE\left[\left(\bar{V}_m^{(1)} - E\left[\bar{V}_m^{(1)}\right]\right)^4\right]\left[1 + 4\sum_{j=1}^{k-1}\sqrt{\tilde{\phi}(j)}\right]. \quad (15)$$

Because $\limsup_n m\delta_m/k < \infty$, then, by combining Eqs. (9), (12), (14), and (15), we have

$$\lim_{n\to\infty}\mathrm{Var}\left(\tilde{S}_n^2\right) = 0. \quad (16)$$

Then, by combining Eqs. (13) and (16), and by Chebyshev's inequality (Durrett [25]), we conclude that $\tilde{S}_n^2$ converges to $\tilde{\sigma}_\infty^2$ in probability as $n \to \infty$. □

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