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Asymptotic representations for importance-sampling estimators of value-at-risk and conditional value-at-risk

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ABSTRACT

Value-at-risk (VaR) and conditional value-at-risk (CVaR) are important risk measures. They are often estimated by using importance-sampling (IS) techniques. In this paper, we derive the asymptotic representations for IS estimators of VaR and CVaR. Based on these representations, we are able to prove the consistency and asymptotic normality of the estimators and to provide simple conditions under which the IS estimators have smaller asymptotic variances than the ordinary Monte Carlo estimators.

1. Introduction

Value-at-risk (VaR) and conditional value-at-risk (CVaR) are two widely used risk measures. They play important roles in investment, risk management, and regulatory control of financial institutions. The Basel Accord II has incorporated the concept of α-VaR, which is defined as the α-quantile of a portfolio value L, and encourages banks to use VaR for daily risk management. The α-CVaR, defined as the average of β-VaR of L for $0 < \beta < \alpha$, has a long history of being used in the insurance industry. It provides information on the potential large losses that an investor may suffer.

Risk managers may consider both of VaR and CVaR at the same time to obtain more information about portfolio risk. There are typically three approaches to estimating them: the variance–covariance approach, the historical simulation approach and the Monte Carlo simulation approach. Among the three, the Monte Carlo simulation approach is frequently used, because it is more general and can be applied to a wider range of risk models. However, the Monte Carlo simulation approach is often time-consuming. In risk management, α is typically close to 0. A large number of replications are needed to obtain accurate estimation of the tail behavior of a loss distribution. Therefore, variance reduction techniques are often used to increase the efficiency of the estimation. Among these techniques, importance sampling (IS) is a natural choice, because it can allocate more samples to the tail of the distribution that is most relevant to the estimation of VaR and CVaR. In this paper, we study the asymptotic properties of the IS estimators of VaR and CVaR and discuss general conditions for IS to be effective.

Because the IS estimators of both VaR and CVaR are rather complicated compared to the typical sample means, we use the method of asymptotic representations to analyze their asymptotic properties. Bahadur [1] used this method to analyze the asymptotic properties of the ordinary estimator of VaR (quantile) by showing that the estimator can be approximated by a sample mean except for a high-order term. Then, the consistency and asymptotic normality of the estimator can be derived easily. In this paper, we derive the asymptotic representations for the IS estimators of both VaR and CVaR, and use them to prove the consistency and asymptotic normality of both estimators. To the best of our knowledge, we are the first to provide such clear representations.

From the asymptotic normality, we give simple conditions on the IS distributions under which the IS scheme is guaranteed to work asymptotically. A good feature of the conditions is that they are same for both VaR and CVaR. Therefore, one can estimate VaR and CVaR simultaneously using the same IS distribution. This feature will greatly help risk managers who consider both risk measures and use them to complement each other.

The literature on applying IS to estimate VaR is growing rapidly. For example, Glynn [7] considered the use of IS for quantile (VaR) estimators; Glasserman et al. [3,4] used IS to estimate the VaR of a portfolio loss for both light-tail and heavy-tail situations; Glasserman and Li [6] applied IS to estimate the VaR of portfolio credit risk; Glasserman and Juneja [5] used IS to estimate the VaR of a sum of independent and identically distributed (i.i.d.) random variables. To the best of our knowledge, however, there is no published work on using IS to estimate CVaR.

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The rest of the paper is organized as follows. Section 2 reviews the IS estimators of VaR and CVaR. In Section 3, we develop asymptotic representations for the IS estimators. From these representations, we can easily prove the consistency and asymptotic normality of the IS estimators. Some lengthy proofs are included in the Appendix.

2. Importance sampling for VaR and CVaR

Let L be a real-valued random variable with a cumulative distribution function (c.d.f.) $F(\cdot)$, and let v and c denote the α-VaR and α-CVaR of L, respectively, for $0 < \alpha < 1$. Then,

$$v = F^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\} \quad \text{and} \quad c = v - \frac{1}{\alpha}\mathrm{E}[v - L]^+,$$

where $x^+ = \max\{x, 0\}$. Note that v is also the α-quantile of L, and $c = \mathrm{E}[L|L \leq v]$ if L has a positive density at v [8]. Under the definitions, we are interested in the left tail of the distribution of L. Therefore, α is often close to 0. Sometimes, v and c are defined for the right tail of L (e.g., [3,4]). We may convert the right tail to the left tail by adding a negative sign to the random variable.

Ordinary Monte Carlo estimation of v and c involves generating n i.i.d. random observations of L, denoted as $L_1, \ldots, L_n$, and estimating them by

$$\tilde{v}_n = \tilde{F}_n^{-1}(\alpha) = \inf\{x : \tilde{F}_n(x) \geq \alpha\}, \tag{1}$$

$$\tilde{c}_n = \tilde{v}_n - \frac{1}{n\alpha}\sum_{i=1}^{n}(\tilde{v}_n - L_i)^+, \tag{2}$$

respectively, where

$$\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{L_i \leq x\} \tag{3}$$

is the empirical distribution of L constructed from $L_1, \ldots, L_n$ and $I\{\cdot\}$ is the indicator function. Note that $\tilde{F}_n(x)$ is an unbiased and consistent estimator of $F(x)$. Serfling [10] and Trindade et al. [11] showed that $\tilde{v}_n$ and $\tilde{c}_n$ are consistent estimators of v and c, respectively, as $n \to \infty$.

Now we introduce the IS estimators of v and c. Suppose we choose an IS distribution function G for which the probability measure associated with G is absolutely continuous with respect to that associated with F, i.e., $F(\mathrm{d}x) = 0$ if $G(\mathrm{d}x) = 0$ for any $x \in \Re$. Let $\mathcal{L}(x) = \frac{F(\mathrm{d}x)}{G(\mathrm{d}x)}$, then $\mathcal{L}$ is called the *likelihood ratio* (LR) function. Then for any $x \in \Re$, we may estimate $F(x)$ by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{L_i \leq x\}\mathcal{L}(L_i). \tag{4}$$

It is easy to see that $F_n(x)$ is also an unbiased and consistent estimator of $F(x)$ as $\tilde{F}_n(x)$ of Eq. (3). Let v_n and c_n denote the IS estimators of v and c. Similar to Eqs. (1) and (2), we define

$$v_n = F_n^{-1}(\alpha) = \inf\{x : F_n(x) \geq \alpha\},$$

$$c_n = v_n - \frac{1}{n\alpha}\sum_{i=1}^{n}(v_n - L_i)^+ \mathcal{L}(L_i).$$

To analyze the asymptotic properties of v_n and c_n, we make the following assumptions.

Assumption 1. There exists an $\epsilon > 0$ such that L has a positive and continuously differentiable density $f(x)$ for any $x \in (v - \epsilon, v + \epsilon)$.

Assumption 1 requires that L has a positive and differentiable density in a neighborhood of v. It implies that $F(v) = \alpha$ and $c = \mathrm{E}[L|L \leq v]$.

Assumption 2. There exist $\varepsilon > 0$ and $C > 0$ such that $\mathcal{L}(x) < C$ for any $x \in (v - \varepsilon, v + \varepsilon)$ and there exists $p > 2$ such that $\mathrm{E}_G[I\{L \leq v + \varepsilon\}\mathcal{L}^p(L)] < \infty$, where E_G denotes the expectation under the IS measure.

Assumption 2 requires the LR function is bounded above in a neighborhood of v and it has a finite $p > 2$ moment on the left tail of L. Effective IS distributions often satisfy $g(x) > f(x)$ for $x < v + \varepsilon$ (see [3–6]), so that they can allocate more samples in the set $\{L < v + \varepsilon\}$ which are most useful in estimating v and c. For these IS distributions, $\mathcal{L}(x) < 1$ for $x < v + \varepsilon$ and, thus, Assumption 2 is satisfied. From Assumption 2 and the positivity of $\mathcal{L}(x)$, we know that for any $\varepsilon' \leq \varepsilon$, $\mathrm{E}_G[I\{L \leq v + \varepsilon'\}\mathcal{L}^2(L)] < \infty$. Therefore, $\mathrm{Var}[I\{L \leq v + \varepsilon'\}\mathcal{L}(L)] < \infty$ for any $\varepsilon' \leq \varepsilon$.

3. Asymptotic representations of the IS estimators

A complicated estimator can often be represented as the sum of several terms whose asymptotic behaviors are clear. This representation is called an *asymptotic representation* of the estimator. Based on the asymptotic representation of an estimator, many asymptotic properties of the estimator, e.g., consistency and asymptotic normality, can be analyzed easily. A famous example is the asymptotic representation of the VaR estimator $\tilde{v}_n$ (also known as Bahadur representation of the quantile estimator). By Bahadur [1], under Assumption 1,

$$\tilde{v}_n = v + \frac{1}{f(v)}\left(\alpha - \frac{1}{n}\sum_{i=1}^{n} I\{L_i \leq v\}\right) + R_n, \tag{5}$$

where $R_n = O_{\mathrm{a.s.}}(n^{-3/4}(\log n)^{3/4})$. The statement $Y_n = O_{\mathrm{a.s.}}(g(n))$ means that $Y_n/g(n)$ is bounded by a constant almost surely. Given the representation, many asymptotic properties of $\tilde{v}_n$ can be analyzed easily. For instance, we may use it to prove the strong consistency and asymptotic normality of $\tilde{v}_n$. By the strong law of large numbers [2], $\frac{1}{n}\sum_{i=1}^{n} I\{L_i \leq v\} \to F(v)$ w.p.1 as $n \to \infty$. Furthermore, because $F(v) = \alpha$ by Assumption 1, it is clear that $\tilde{v}_n \to v$ w.p.1 as $n \to \infty$. Thus, $\tilde{v}_n$ is a strongly consistent estimator of v. Similarly, by the central limit theorem [2],

$$\sqrt{n}\left(\alpha - \frac{1}{n}\sum_{i=1}^{n} I\{L_i \leq v\}\right) \Rightarrow \sqrt{\alpha(1-\alpha)}N(0, 1) \quad \text{as } n \to \infty,$$

where "$\Rightarrow$" denotes "converge in distribution" and $N(0, 1)$ denotes a standard normal random variable. Then,

$$\sqrt{n}(\tilde{v}_n - v) \Rightarrow \frac{\sqrt{\alpha(1-\alpha)}}{f(v)}N(0, 1) \quad \text{as } n \to \infty. \tag{6}$$

Therefore, $\tilde{v}_n$ is asymptotically normally distributed.

In the rest of this section, we develop asymptotic representations of the IS estimators v_n and c_n, and use them to analyze the consistency and asymptotic normality of v_n and c_n.

3.1. Asymptotic representation of v_n

We first consider the asymptotic representation of v_n. Note that by Taylor expansion [12, p. 110], $F(v_n) - F(v) = f(v)(v_n - v) - A_{1,n}$, where $A_{1,n}$ is the remainder term. Then, we have

$$v_n = v + \frac{F(v_n) - F(v)}{f(v)} + \frac{1}{f(v)}A_{1,n}. \tag{7}$$

Let $A_{2,n} = F(v_n) + F_n(v) - F_n(v_n) - F(v)$ and $A_{3,n} = F_n(v_n) - F(v)$. It is easy to see that

$$F(v_n) - F(v) = F(v) - F_n(v) + A_{2,n} + A_{3,n}.$$

Therefore, by Eq. (7), we have

$$v_n = v + \frac{F(v) - F_n(v)}{f(v)} + \frac{A_{1,n} + A_{2,n} + A_{3,n}}{f(v)}. \tag{8}$$

In the following lemma, we provide the orders of $A_{1,n}$, $A_{2,n}$ and $A_{3,n}$. The proof of the lemma is included in the Appendix. In the lemma, we use the statement $U_n = o_p(g(n))$ which means $U_n/g(n) \to 0$ in probability as $n \to \infty$.

Lemma 1. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied. Then, $A_{1,n} = O_{a.s.}(n^{-1+2/p+\delta})$, $A_{2,n} = O_{a.s.}(n^{-3/4+1/(2p)+\delta})$, $A_{3,n} = O_{a.s.}(n^{-1})$ for any $\delta > 0$, and $A_{1,n} = o_p(n^{-1/2})$, $A_{2,n} = o_p(n^{-1/2})$, $A_{3,n} = o_p(n^{-1/2})$. Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $A_{1,n} = O_{a.s.}(n^{-1}\log n)$, $A_{2,n} = O_{a.s.}(n^{-3/4}(\log n)^{3/4})$ and $A_{3,n} = O_{a.s.}(n^{-1})$.*

Let $\mathcal{L}_i$ denote $\mathcal{L}(L_i)$ for all $i = 1,\ldots,n$. By Lemma 1, we can prove the following theorem on the asymptotic representation of v_n.

Theorem 1. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied. Then,*

$$v_n = v + \frac{1}{f(v)}\left(\alpha - \frac{1}{n}\sum_{i=1}^{n} I\{L_i \le v\}\mathcal{L}_i\right) + A_n,$$

where $A_n = o_p(n^{-1/2})$ and $A_n = O_{a.s.}(t(n,\delta))$ with $t(n,\delta) = \max\{n^{-1+2/p+\delta}, n^{-3/4+1/(2p)+\delta}\}$ for any $\delta > 0$. Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $A_n = O_{a.s.}(n^{-3/4}(\log n)^{3/4})$.

Proof. By Assumption 1, $F(v) = \alpha$. Then, by Eq. (4), $F(v) - F_n(v) = \alpha - \frac{1}{n}\sum_{i=1}^{n} I\{L_i \le v\}\mathcal{L}_i$. Let $A_n = \frac{1}{f(v)}(A_{1,n} + A_{2,n} + A_{3,n})$. Since $f(v) > 0$ by Assumption 1, the conclusions of the theorem follow directly from Eq. (8) and Lemma 1. □

Because $\tilde{v}_n$ is a special case of v_n where $\mathcal{L}(x) = 1$ for all $x \in \Re$, Bahadur representation of Eq. (5) may be viewed as a special case of Theorem 1.

Let Var_G denote the variance under the IS measure. From Theorem 1, it is also straight-forward to prove the following corollary on the strong consistency and asymptotic normality of v_n.

Corollary 1. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied. Then, as $n \to \infty$, $v_n \to v$ w.p.1 and*

$$\sqrt{n}(v_n - v) \Rightarrow \frac{\sqrt{\mathrm{Var}_G\left[I\{L \le v\}\mathcal{L}(L)\right]}}{f(v)} N(0,1).$$

Remark 1. The conclusions of Corollary 1 have also been proved by Glynn [7] under Assumption 1 and the assumption that $\mathrm{E}_G[\mathcal{L}^3(L)] < \infty$.

Note that

$$\begin{aligned}\mathrm{Var}_G\left[I\{L \le v\}\mathcal{L}(L)\right] &= \mathrm{E}_G\left[I\{L \le v\}\mathcal{L}^2(L)\right] - \mathrm{E}_G^2\left[I\{L \le v\}\mathcal{L}(L)\right] \\ &= \mathrm{E}\left[I\{L \le v\}\mathcal{L}(L)\right] - \alpha^2. \qquad (9)\end{aligned}$$

If $\mathcal{L}(x) < 1$ for all $x \le v$ as in [3–6], by Eq. (9), $\mathrm{Var}_G[I\{L \le v\}\mathcal{L}(L)] < \alpha(1-\alpha)$. Compared to Eq. (6), the IS estimator v_n has a smaller asymptotic variance than the ordinary estimator $\tilde{v}_n$. Therefore, if the IS distribution is selected appropriately, it may improve the efficiency of VaR estimation.

3.2. Asymptotic representation of c_n

We now consider the asymptotic representation of c_n. Note that

$$\begin{aligned}c_n &= v_n - \frac{1}{n\alpha}\sum_{i=1}^{n}(v_n - L_i)^+\mathcal{L}_i = v - \frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)^+\mathcal{L}_i \\ &\quad + (v_n - v) - \frac{1}{n\alpha}\sum_{i=1}^{n}\left[(v_n - L_i)^+ - (v - L_i)^+\right]\mathcal{L}_i.\end{aligned}$$

Furthermore, note that

$$\begin{aligned}&\frac{1}{n\alpha}\sum_{i=1}^{n}\left[(v_n - L_i)^+ - (v - L_i)^+\right]\mathcal{L}_i \\ &= \frac{1}{n\alpha}\sum_{i=1}^{n}\left[(v_n - L_i)I\{L_i \le v_n\} - (v - L_i)I\{L_i \le v\}\right]\mathcal{L}_i \\ &= \frac{1}{n\alpha}\sum_{i=1}^{n}\left[(v_n - v)I\{L_i \le v_n\}\right]\mathcal{L}_i \\ &\quad + \frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)\left[I\{L_i \le v_n\} - I\{L_i \le v\}\right]\mathcal{L}_i.\end{aligned}$$

Then,

$$\begin{aligned}c_n &= v - \frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)^+\mathcal{L}_i + \frac{1}{\alpha}(v_n - v)\left(\alpha - F_n(v_n)\right) \\ &\quad - \frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)\left[I\{L_i \le v_n\} - I\{L_i \le v\}\right]\mathcal{L}_i. \qquad (10)\end{aligned}$$

Since

$$\begin{aligned}&\left|\frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)\left[I\{L_i \le v_n\} - I\{L_i \le v\}\right]\mathcal{L}_i\right| \\ &\le \frac{1}{\alpha}|v_n - v| \cdot |F_n(v_n) - F_n(v)|, \qquad (11)\end{aligned}$$

by Eqs. (10) and (11), we have

$$c_n = v - \frac{1}{n\alpha}\sum_{i=1}^{n}(v - L_i)^+\mathcal{L}_i + B_n, \qquad (12)$$

where

$$|B_n| \le \frac{1}{\alpha}|v_n - v|\left(2|F_n(v_n) - F(v)| + |F_n(v) - F(v)|\right).$$

In the following lemma, we prove the order of B_n. The proof of the lemma is included in the Appendix.

Lemma 2. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied. Then, $B_n = O_{a.s.}(n^{-1+2/p+\delta})$ and $B_n = o_p(n^{-1/2})$. Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $B_n = O_{a.s.}(n^{-1}\log n)$.*

Then, we have the following theorem on the asymptotic representation of c_n. Note that the conclusion of the theorem follows directly from Eq. (12) and Lemma 2. Therefore, we omit the proof.

Theorem 2. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied. Then,*

$$c_n = c + \left(\frac{1}{n}\sum_{i=1}^{n}\left[v - \frac{1}{\alpha}(v - L_i)^+\mathcal{L}_i\right] - c\right) + B_n,$$

where $B_n = O_{a.s.}(n^{-1+2/p+\delta})$ for any $\delta > 0$ and $B_n = o_p(n^{-1/2})$. Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $B_n = O_{a.s.}(n^{-1}\log n)$.

Note that $\mathrm{E}_G\left[v - \frac{1}{\alpha}(v - L)^+\mathcal{L}(L)\right] = c$. Then, by the strong law of large numbers and the central limit theorem, it is straight-forward to prove the following corollary on the strong consistency and asymptotic normality of c_n.

Corollary 2. *For a fixed $\alpha \in (0,1)$, suppose that Assumptions 1 and 2 are satisfied and $\mathrm{E}_G[(v-L)^2\mathcal{L}(L)^2 I\{L < v\}] < \infty$. Then, as $n \to \infty$, $c_n \to c$ w.p.1 and*

$$\sqrt{n}(c_n - c) \Rightarrow \frac{\sqrt{\mathrm{Var}_G\left[(v - L)^+\mathcal{L}(L)\right]}}{\alpha} N(0,1).$$

Furthermore, we may set $\mathcal{L}(x) = 1$ for all $x \in \Re$. Then, the conclusions of Theorem 2 and Corollary 2 apply to $\tilde{c}_n$, the ordinary Monte Carlo estimator of c. We then have the following corollary.

Corollary 3. *For a fixed $\alpha \in (0,1)$, suppose that Assumption 1 is satisfied. Then,*

$$\tilde{c}_n = c + \left(\frac{1}{n}\sum_{i=1}^{n}\left[v - \frac{1}{\alpha}(v - L_i)^+\right] - c\right) + C_n,$$

where $C_n = O_{a.s.}(n^{-1}\log n)$, *and* $\tilde{c}_n \to c$ *w.p.*1 *as* $n \to \infty$. *Furthermore, if* $\mathrm{E}[(v-L)^2 I\{L<v\}] < \infty$, *then*

$$\sqrt{n}(\tilde{c}_n - c) \Rightarrow \frac{\sqrt{\mathrm{Var}[(v-L)^+]}}{\alpha} N(0,1) \quad \text{as } n \to \infty.$$

Remark 2. The strong consistency and asymptotic normality of $\tilde{c}_n$ have also been studied by a number of papers in the literature, including [11,8], using different methods.

If the IS distribution satisfies $\mathcal{L}(x) < 1$ for all $x \le v$, as in [3–6], it is easy to show that $\mathrm{Var}_G[(v-L)^+\mathcal{L}(L)] < \mathrm{Var}[(v-L)^+]$. Therefore, by Corollaries 2 and 3, the IS estimator c_n has a smaller asymptotic variance than the ordinary estimator $\tilde{c}_n$. Note that the same condition can also reduce the asymptotic variance of the IS estimator of VaR as shown in Section 3.1. Therefore, if the IS distribution is selected appropriately, it may improve the efficiency of VaR and CVaR estimations at the same time.

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Appendix

A.1. Proof of Lemma 1

A similar result has been proved by Serfling [10] for $\tilde{v}_n$. In this section, we mainly follow his steps. However, we need to handle the likelihood-ratio term that does not appear in $\tilde{v}_n$.

A.1.1. Three propositions

Proposition 1. *For a fixed* $\alpha \in (0,1)$, *suppose that Assumption 2 is satisfied. Then for any* $\gamma < \varepsilon$,

$$\Pr\{|v_n - v| > \gamma\} \le C_p n^{-p/2}\delta_\gamma^{-p}$$

for sufficiently large n, where $\delta_\gamma = \min\{F(v+\gamma)-\alpha, \alpha - F(v-\gamma)\}$, *and* C_p *is a constant related to p. Furthermore, if* $\mathcal{L}(x) < C$ *for any* $x \in (-\infty, v+\varepsilon)$, *then*

$$\Pr\{|v_n - v| > \gamma\} \le 2\mathrm{e}^{-2n\delta_\gamma^2/(C+1)^2}.$$

Proof. Note that

$$\Pr\{|v_n - v| > \gamma\} = \Pr\{v_n > v+\gamma\} + \Pr\{v_n < v-\gamma\}.$$

Because $v_n = F_n^{-1}(\alpha) = \inf\{x : F_n(x) \ge \alpha\}$, $v_n > v+\gamma$ and $v_n \le v-\gamma$ are equivalent to $F_n(v+\gamma) < \alpha$ and $F_n(v-\gamma) \ge \alpha$, respectively. We have

$$\begin{aligned}\Pr\{|v_n - v| > \gamma\} &\le \Pr\{F_n(v+\gamma) < \alpha\} + \Pr\{F_n(v-\gamma) \ge \alpha\} \\ &\le \Pr\{F(v+\gamma) - F_n(v+\gamma) > F(v+\gamma) - \alpha\} \\ &\quad + \Pr\{F_n(v-\gamma) - F(v-\gamma) \ge \alpha - F(v-\gamma)\}. \qquad (13)\end{aligned}$$

Moreover,

$$\begin{aligned}&\Pr\{F(v+\gamma) - F_n(v+\gamma) > F(v+\gamma) - \alpha\} \\ &= \Pr\left\{\sum_{i=1}^n [F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i] > n(F(v+\gamma)-\alpha)\right\},\end{aligned}$$

and

$$\begin{aligned}&\Pr\{F_n(v-\gamma) - F(v-\gamma) \ge \alpha - F(v-\gamma)\} \\ &= \Pr\left\{\sum_{i=1}^n [I\{L_i \le v-\gamma\}\mathcal{L}_i - F(v-\gamma)] \ge n(\alpha - F(v-\gamma))\right\}.\end{aligned}$$

Note that, when Assumption 2 is satisfied, combining with Markov's inequality [2, p. 14], we have

$$\begin{aligned}&\Pr\left\{\sum_{i=1}^n F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i > n(F(v+\gamma)-\alpha)\right\} \\ &\le \frac{\mathrm{E}\left[\left|\sum_{i=1}^n F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i\right|^p\right]}{n^p(F(v+\gamma)-\alpha)^p}.\end{aligned}$$

By Rosenthal's inequality [9], we know

$$\begin{aligned}&\mathrm{E}\left[\left|\sum_{i=1}^n F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i\right|^p\right] \\ &\le \tilde{C}_p \max\left\{\sum_{i=1}^n \mathrm{E}|F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i|^p,\right. \\ &\qquad \left.\left(\sum_{i=1}^n \mathrm{E}\big[F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i\big]^2\right)^{p/2}\right\},\end{aligned}$$

where $\tilde{C}_p = 2\max\left\{p^p, p^{p/2+1}\mathrm{e}^p\int_0^\infty x^{p/2-1}(1-x)^{-p}\mathrm{d}x\right\}$. When n is sufficiently large,

$$\begin{aligned}&\sum_{i=1}^n \mathrm{E}|F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i|^p \\ &\le \left(\sum_{i=1}^n \mathrm{E}[F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i]^2\right)^{p/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Pr\left\{\sum_{i=1}^n [F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i] > n(F(v+\gamma)-\alpha)\right\} \\ &\le \tilde{C}_p \frac{\left(\mathrm{E}\,[F(v+\gamma) - I\{L_1 \le v+\gamma\}\mathcal{L}_1]^2\right)^{p/2}}{n^{p/2}(F(v+\gamma)-\alpha)^p}.\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}&\Pr\left\{\sum_{i=1}^n [I\{L_i \le v-\gamma\}\mathcal{L}_i - F(v-\gamma)] \ge n(\alpha - F(v-\gamma))\right\} \\ &\le \tilde{C}_p \frac{\left(\mathrm{E}\,[I\{L_1 \le v-\gamma\}\mathcal{L}_1 - F(v-\gamma)]^2\right)^{p/2}}{n^{p/2}(\alpha - F(v-\gamma))^p}.\end{aligned}$$

Let

$$\begin{aligned}C_p = 2\tilde{C}_p \max\Big\{&\left(\mathrm{E}\,[F(v+\gamma) - I\{L_1 \le v+\gamma\}\mathcal{L}_1]^2\right)^{p/2}, \\ &\left(\mathrm{E}\,[I\{L_1 \le v-\gamma\}\mathcal{L}_1 - F(v-\gamma)]^2\right)^{p/2}\Big\}.\end{aligned}$$

Combing with Eq. (13), we have, for sufficiently large n,

$$\Pr\{|v_n - v| > \gamma\} \le C_p n^{-p/2}\delta_\gamma^{-p}.$$

If we further have $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, both $F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i$ and $I\{L_i \le v-\gamma\}\mathcal{L}_i - F(v-\gamma)$ are bounded. Thus, we can apply Hoeffding's inequality [10, p. 75], to have

$$\begin{aligned}&\Pr\left\{\sum_{i=1}^n [F(v+\gamma) - I\{L_i \le v+\gamma\}\mathcal{L}_i] > n(F(v+\gamma)-\alpha)\right\} \\ &\le \exp\left\{-\frac{2n(F(v+\gamma)-\alpha)^2}{(C+1)^2}\right\},\end{aligned}$$

$$\Pr\left\{\sum_{i=1}^{n}[I\{L_i \le v-\gamma\}\mathcal{L}_i - F(v-\gamma)] \ge n(\alpha - F(v-\gamma))\right\}$$
$$\le \exp\left\{-\frac{2n(\alpha - F(v-\gamma))^2}{(C+1)^2}\right\}.$$

Then, $\Pr(|v_n - v| > \gamma) \le 2e^{-2n\delta_\gamma^2/(C+1)^2}$. This completes the proof of the proposition. □

Proposition 2. *Let $\epsilon_{n,\delta} = \frac{2}{f(v)} n^{-1/2+1/p+\delta}$ with $\delta > 0$. For a fixed $\alpha \in (0,1)$, suppose Assumptions 1 and 2 are satisfied. Then, $|v_n - v| = O_{a.s.}(\epsilon_{n,\delta})$ for any $\delta > 0$ and $|v_n - v| = o_p(n^{-1/2}g(n))$ for any function $g(n) \to \infty$ as $n \to \infty$. Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $|v_n - v| = O_{a.s.}(\epsilon_n)$ with $\epsilon_n = \frac{2C}{f(v)} n^{-1/2}(\log n)^{1/2}$.*

Proof. By Assumption 1, v is a unique solution to $F(x) = \alpha$. Note that

$$F(v+\epsilon_{n,\delta}) - \alpha = F(v+\epsilon_{n,\delta}) - F(v) = f(v)\epsilon_{n,\delta} + o(\epsilon_{n,\delta}).$$

When n is sufficiently large and δ is sufficiently small, $\epsilon_{n,\delta} < \epsilon$, where ϵ is defined in Assumption 2. Then, $F(v+\epsilon_{n,\delta}) - \alpha \ge f(v)\epsilon_{n,\delta}/2$. Similarly, we can also prove that $\alpha - F(v - \epsilon_{n,\delta}) \ge f(v)\epsilon_{n,\delta}/2$. Therefore,

$$\delta_{\epsilon_{n,\delta}} = \min\{F(v+\epsilon_{n,\delta}) - \alpha, \alpha - F(v-\epsilon_{n,\delta})\}$$
$$\ge \frac{f(v)}{2}\epsilon_{n,\delta} \ge n^{-\frac{1}{2}+\frac{1}{p}+\delta}.$$

Hence, by Proposition 1, for n sufficiently large, $\Pr\{|v_n - v| > \epsilon_{n,\delta}\} \le C_p n^{-1-p\delta}$, where C_p is defined in Proposition 1. Because $\sum_{n=1}^{\infty} C_p n^{-1-p\delta} < \infty$ for any $\delta > 0$, by Borel–Cantelli Lemma [2, p. 46], we have $|v_n - v| = O_{a.s.}(\epsilon_{n,\delta})$. Similarly, for any $\alpha > 0$, we have

$$\Pr\left\{\frac{|v_n - v|}{n^{-1/2}g(n)} > \alpha\right\} = \Pr\left\{|v_n - v| > \alpha n^{-1/2}g(n)\right\}$$
$$\le \overline{C}(g(n))^{-p},$$

where $\overline{C}$ is a constant. Then, $|v_n - v| = o_p(n^{-1/2}g(n))$.

Furthermore, if $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, by the same approach, we can prove $|v_n - v| = O_{a.s.}(\epsilon_n)$ with $\epsilon_n = \frac{2C}{f(v)} n^{-1/2}(\log n)^{1/2}$. This completes the proof of the proposition. □

Proposition 3. *Let $T_n = \sup_{|x|\le\epsilon_{n,\delta}} |F_n(v+x) - F_n(v) - F(v+x) + F(v)|$, where $\epsilon_{n,\delta}$ is defined in Proposition 2. Suppose Assumptions 1 and 2 are satisfied. Then, $T_n = O_{a.s.}(n^{-3/4+1/(2p)+\delta})$ for any $\delta > 0$.*

Furthermore, let $K_n = \sup_{|x|\le\epsilon_n} |F_n(v+x) - F_n(v) - F(v+x) + F(v)|$, where ϵ_n is defined in Proposition 2. If $\mathcal{L}(x) < C$ for any $x \in (-\infty, v+\varepsilon)$, then $K_n = O_{a.s.}(n^{-3/4}(\log n)^{3/4})$.

Proof. We have $\epsilon_{n,\delta} < \min\{\epsilon, \varepsilon\}$ for n sufficiently large and δ sufficiently small, where ϵ and ε are defined in Assumptions 1 and 2, respectively, and $\epsilon_{n,\delta}$ is defined in Proposition 2. Let $b_n = \left\lceil \frac{2}{f(v)} n^{1/4+1/p+\delta} \right\rceil$. For any integer $l \in [-b_n, b_n]$, we let $\xi_{l,n} = v + l\epsilon_{n,\delta}/b_n$. For any $|x| < \epsilon_{n,\delta}$, we can find l such that $v + x \in [\xi_{l,n}, \xi_{l+1,n})$. Then,

$$F_n(\xi_{l,n}) - F(\xi_{l+1,n}) \le F_n(v+x) - F(v+x) \le F_n(\xi_{l+1,n}) - F(\xi_{l,n}),$$

which is equivalent to

$$F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v) + F(\xi_{l,n}) - F(\xi_{l+1,n})$$
$$\le F_n(v+x) - F_n(v) - F(v+x) + F(v)$$
$$\le F_n(\xi_{l+1,n}) - F_n(v) - F(\xi_{l+1,n}) + F(v) + F(\xi_{l+1,n}) - F(\xi_{l,n}).$$

Then, we have

$$T_n \le \sup_{l\in[-b_n,b_n]} \left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| + \sup_{l\in[-b_n,b_n-1]} \left|F(\xi_{l+1,n}) - F(\xi_{l,n})\right|. \quad (14)$$

Note that $|F(\xi_{l+1,n}) - F(\xi_{l,n})| = f(z)\epsilon_{n,\delta}/b_n$ for some $z \in (\xi_{l,n}, \xi_{l+1,n})$. Let $\tilde{f} = \sup_{|x|\le\epsilon} f(v+x)$. Then,

$$\sup_{l\in[-b_n,b_n-1]} \left|F(\xi_{l+1,n}) - F(\xi_{l,n})\right| \le \tilde{f} n^{-3/4}.$$

From Assumption 2, we know $\mathcal{L}(x) < C$ for any $x \in (v - \epsilon_{n,\delta}, v + \epsilon_{n,\delta})$. Let $\xi_{n,\delta} = n^{-3/4+1/(2p)+\delta/2}(\log n)^{1/2}$.

By Bernstein's inequality [10, p. 95], for any $c_1 > 0$

$$\Pr\left\{\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,\delta}\right\}$$
$$\le 2\exp\left\{-\frac{3c_1^2 n\xi_{n,\delta}^2}{6\sigma^2 + 2Cc_1\xi_{n,\delta}}\right\}, \quad (15)$$

where $\sigma^2 = \mathrm{Var}[I\{L_i \in (v, \xi_{l,n}]\}\mathcal{L}_i] \le \mathrm{E}[I\{L_i \in (v, \xi_{l,n}]\}\mathcal{L}_i^2] \le \mathrm{E}[I\{L_i \in (v, \xi_{l,n}]\}]C^2 \le C^2\tilde{f}\epsilon_{n,\delta}$.

Inputting the upper bound of σ^2 into Eq. (15), we have

$$\Pr\left\{\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,\delta}\right\}$$
$$\le 2\exp\left\{-\frac{3c_1^2 n^{-\frac{1}{2}+\frac{1}{p}+\delta}\log n}{6C^2\tilde{f}\frac{2}{f(v)}n^{-\frac{1}{2}+\frac{1}{p}+\delta} + 2Cc_1 n^{-\frac{3}{4}+\frac{1}{2p}+\frac{\delta}{2}}(\log n)^{1/2}}\right\}$$
$$\le 2\exp\left\{-\frac{3c_1^2\log n}{6C^2\tilde{f}\frac{2}{f(v)} + 2Cc_1 n^{-\frac{1}{4}-\frac{1}{2p}-\frac{\delta}{2}}(\log n)^{1/2}}\right\}.$$

When n is sufficiently large, we can choose c_1 big enough such that

$$\frac{3c_1^2}{6C^2\cdot\tilde{f}\frac{2}{f(v)} + 2Cc_1 n^{-\frac{1}{4}-\frac{1}{2p}-\frac{\delta}{2}}(\log n)^{1/2}} > 2.$$

Then, $\Pr\{|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)| > c_1\xi_{n,\delta}\} \le 2n^{-2}$. Therefore, when $\delta < 1/4$,

$$\Pr\left\{\sup_{l\in[-b_n,b_n]}\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,\delta}\right\}$$
$$\le \sum_{l=-b_n}^{b_n}\Pr\left\{\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,\delta}\right\}$$
$$\le 2\left\lceil\frac{2}{f(v)}n^{1/4+1/p+\delta}\right\rceil n^{-2}$$

and

$$\sum_{n=1}^{\infty}\Pr\left\{\sup_{l\in[-b_n,b_n]}\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,c}\right\}$$
$$\le \sum_{n=1}^{\infty}2\left\lceil\frac{2}{f(v)}n^{1/4+1/p+\delta}\right\rceil n^{-2} < \infty. \quad (16)$$

Note that

$$\sum_{n=1}^{\infty}\Pr\left\{T_n > c_1\,\xi_{n,\delta}\right\}$$
$$\le \sum_{n=1}^{\infty}\Pr\left\{\sup_{l\in[-b_n,b_n]}\left|F_n(\xi_{l,n}) - F_n(v) - F(\xi_{l,n}) + F(v)\right| > c_1\xi_{n,\delta}\right\}$$
$$+ \sum_{n=1}^{\infty}\Pr\left\{\sup_{l\in[-b_n,b_n]}\left|F(\xi_{l+1,n}) - F(\xi_{l,n})\right| > c_1\xi_{n,\delta}\right\}.$$

The event $\sup_{l\in[-b_n,b_n-1]}|F(\xi_{l+1,n})-F(\xi_{l,n})|>c_1\xi_{n,\delta}$ is actually a deterministic event. From Eq. (14), we know for n big enough this event will never happen and thus the probability is 0. Combining with Eq. (16) and Borel–Cantelli Lemma, we have $T_n=O_{\text{a.s.}}(\xi_{n,\delta})=O_{\text{a.s.}}\left(n^{-\frac{3}{4}+\frac{1}{2p}+\delta}(\log n)^{1/2}\right)$. Note that $\log n=O(n^\delta)$, then we have $T_n=O_{\text{a.s.}}\left(n^{-\frac{3}{4}+\frac{1}{2p}+\delta}\right)$.

Furthermore, if $\mathcal{L}(x)<C$ for any $x\in(-\infty,v+\varepsilon)$, we can similarly prove $K_n=O_{\text{a.s.}}(n^{-3/4}(\log n)^{3/4})$. This completes the proof of the proposition. □

A.1.2. Proof of Lemma 1

Proof. First, we prove, $A_{1,n}=O_{\text{a.s.}}(n^{-1+2/p+\delta})$ and $A_{1,n}=o_p(n^{-1/2})$. Note that

$$\begin{aligned}\Pr\{A_{1,n}>A\}&\le\Pr\{A_{1,n}>A,|v_n-v|\le\epsilon_{n,\delta}\}\\&\quad+\Pr\{A_{1,n}>A,|v_n-v|>\epsilon_{n,\delta}\}\\&\le\Pr\{A_{1,n}>A,|v_n-v|\le\epsilon_{n,\delta}\}\\&\quad+\Pr\{|v_n-v|>\epsilon_{n,\delta}\}.\end{aligned}$$

From Assumption 1 and a second-order Taylor expansion, we know that, when $|v_n-v|<\epsilon$, $A_{1,n}<M(v_n-v)^2$ with $M=\sup_{s\in(v-\epsilon,v+\epsilon)}f'(s)$. When n is sufficiently large and δ is sufficiently small, $\epsilon_{n,\delta}<\epsilon$. Let $A=M\epsilon_{n,\delta}^2$. Then, combining with Proposition 2, we have $A_{1,n}=O_{\text{a.s.}}(\epsilon_{n,\delta}^2)=O_{\text{a.s.}}(n^{-1+2/p+\delta})$ for any $\delta>0$. Let $A=n^{-1}g(n)$, combining with Proposition 2, we have $\Pr(A_{1,n}>cn^{-1})\to0$ as $n\to\infty$ for any $c>0$. Thus, $A_{1,n}=o_p(n^{-1/2})$.

Second, we prove $A_{2,n}=O_{\text{a.s.}}\left(n^{-\frac{3}{4}+\frac{1}{2p}+\delta}\right)$. With $\epsilon_{n,\delta}$ defined in Proposition 2 and c_1 and $\xi_{n,\delta}$ defined in the proof of Proposition 3, we have

$$\begin{aligned}&\Pr\left\{\left|A_{2,n}\right|>c_1\xi_{n,\delta}\right\}\\&=\Pr\left\{|F_n(v_n)-F(v_n)+F(v)-F_n(v)|>c_1\xi_{n,\delta}\right\}\\&=\Pr\{|F_n(v_n)-F(v_n)+F(v)-F_n(v)|\\&\quad>c_1\xi_{n,\delta},|v_n-v|>\epsilon_{n,\delta}\}\\&\quad+\Pr\{|F_n(v_n)-F(v_n)+F(v)-F_n(v)|\\&\quad>c_1\xi_{n,\delta},\ |v_n-v|\le\epsilon_{n,\delta}\}\\&\le\Pr\left\{|v_n-v|>\epsilon_{n,\delta}\right\}+\Pr\left\{T_n>c_1\xi_{n,\delta},|v_n-v|\le\epsilon_{n,\delta}\right\}.\end{aligned}$$

Combining with Propositions 2 and 3, we can easily see $A_{2,n}=O_{\text{a.s.}}(\xi_{n,\delta})=O_{\text{a.s.}}\left(n^{-\frac{3}{4}+\frac{1}{2p}+\delta}(\log n)^{1/2}\right)$ for $\delta>0$. Moreover, we have $n^{1/2}A_{2,n}=O_{\text{a.s.}}\left(n^{-\frac{1}{4}+\frac{1}{2p}+\delta}(\log n)^{1/2}\right)$. For δ sufficiently small, we know $n^{1/2}A_{2,n}\to0$ w.p.1 and thus as $n\to0$, $n^{1/2}A_{2,n}\to0$ in probability.

Then, we prove $A_{3,n}=O_{\text{a.s.}}(n^{-1})$ and $A_{3,n}=o_p(n^{-1/2})$. Let $\underline{v}_n=\max\{L_i,i=1,\ldots,n:F_n(L_i)<\alpha\}$. Then, we have $F_n(\underline{v}_n)<\alpha$ and $F_n(v_n)=F_n(\underline{v}_n)+\mathcal{L}(v_n)/n$.

$$\begin{aligned}\Pr\left\{A_{3,n}>C/n\right\}&=\Pr\left\{F_n(v_n)>\alpha+C/n,|v_n-v|\le\epsilon_{n,\delta}\right\}\\&\quad+\Pr\left\{F_n(v_n)>\alpha+C/n,|v_n-v|>\epsilon_{n,\delta}\right\}\\&\le\Pr\left\{F_n(v_n)>\alpha+C/n,|v_n-v|\le\epsilon_{n,\delta}\right\}\\&\quad+\Pr\left\{|v_n-v|>\epsilon_{n,\delta}\right\}.\end{aligned}$$

For n big enough and δ sufficient small, we have $\epsilon_{n,\delta}<\varepsilon$ with ε and C defined in Assumption 2. When $|v_n-v|<\varepsilon$, $F_n(v_n)=F_n(\underline{v}_n)+\mathcal{L}(v_n)/n<\alpha+C/n$. Thus the first part of the equation is 0 for n big enough. Combining with Proposition 2, we have $A_{3,n}=O_{\text{a.s.}}(n^{-1})$ and therefore $A_{3,n}=o_p(n^{-1/2})$.

Furthermore, if $\mathcal{L}(x)<C$ for any $x\in(-\infty,v+\varepsilon)$, we can similarly prove that $A_{1,n}=O_{\text{a.s.}}(n^{-1}\log n)$, $A_{2,n}=O_{\text{a.s.}}(n^{-3/4}(\log n)^{3/4})$ and $A_{3,n}=O_{\text{a.s.}}(n^{-1})$. This completes the proof of the lemma. □

A.2. Proof of Lemma 2

Proof. By Lemma 1, we know $F_n(v_n)-F(v)=A_{3,n}=O_{\text{a.s.}}(n^{-1})$ and $A_{3,n}=o_p(n^{-1/2})$. From Lemma 1, $v_n-v=O_{\text{a.s.}}(n^{-1/2+1/p+\delta})$ and from Proposition 2, $v_n-v=o_p(n^{-1/2}g(n))$ with $g(n)$ defined in Proposition 2. It suffices to prove that $F_n(v)-\alpha=O_{\text{a.s.}}(n^{-1/2+1/p+\delta})$ and $F_n(v)-\alpha=o_p(n^{-1/2}g(n))$. Similar arguments as in Proposition 1 yield that, for n sufficiently large

$$\begin{aligned}\Pr\{|F_n(v)-\alpha|>\gamma\}&=\Pr\left\{\left|\sum_{i=1}^n(I\{L_i\le v\}\mathcal{L}_i-\alpha)\right|>n\gamma\right\}\\&\le\frac{\mathrm{E}\left[\left|\sum_{i=1}^n(I\{L_i\le v\}\mathcal{L}_i-\alpha)\right|^p\right]}{(n\gamma)^p}<\overline{C}_p\frac{1}{n^{p/2}(\gamma)^p},\end{aligned}$$

where $\overline{C}_p$ is a constant. For any $c>0$, let $\gamma=cn^{-1/2}g(n)$. We have

$$\Pr\left\{\frac{|F_n(v)-\alpha|}{n^{-1/2}g(n)}>c\right\}=\Pr\{|F_n(v)-\alpha|>\gamma\}<\overline{C}_p/[cg(n)]^p.$$

Thus, $|F_n(v)-\alpha|=o_p(n^{-1/2}g(n))$. Similarly, combining with Borel–Cantelli Lemma, we can prove $|F_n(v)-\alpha|=O_{\text{a.s.}}\left(n^{-\frac{1}{2}+\frac{1}{p}+\delta}\right)$. When $\mathcal{L}(x)<C$ for any $x\in(-\infty,v+\varepsilon)$, we can similarly prove $B_n=O_{\text{a.s.}}(n^{-1}\log n)$. This completes the proof of the lemma. □

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