Finite-time regret analysis of Kierfer-Wolfowitz stochastic approximation algorithm and nonparametric multi-product dynamic pricing with unknown demand

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Contents

PART 1

“Robust” Dynamic Pricing

The “price of misspecification” is expected to be significant if the parametric model is overly restrictive. Somewhat surprisingly, we show (under reasonably general conditions) that this need not be the case.

1.1 Dynamic Pricing

- Demand: $D_t = \lambda(p_t) + \varepsilon_t, \quad t = 1, 2, 3, \ldots$

  where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic (decreasing) function;

  $\varepsilon_t, t \geq 1$ are zero-mean i.i.d random variables.

- Regret: $R(\pi, T) = p^* \lambda(p^*) T - E^\pi (\sum_{t=1}^{T} p_t D_t)$

- Parametric Model: Misspecification:

1. 2 "Robust" linear model

- Besbes and Zeevi (2015):

Semimyopic pricing scheme: $\hat{\pi}(\hat{p}_1, \{I_i, \delta_i, \mathcal{T}_i: i \geq 1\})$
Set $t_1 = 0$.
For $i \geq 1$:
Step 1. Pricing and information collection
Set prices

$$p_t = \hat{p}_i, \quad t = t+1, \ldots, t+I_i,$$
$$p_t = \hat{p}_i + \delta_i, \quad t = t+I_i+1, \ldots, t+2I_i.$$

Set $t_{i+1} = t_i + 2I_i$.
Step 2. Recalibration

$$(\hat{\alpha}_{i+1}, \hat{\beta}_{i+1}) = \arg\min_{\alpha,\beta} \left\{ \sum_{t \in \mathcal{T}_i} [D_t - (\alpha - \beta p_t)]^2 \right\}. \quad (5)$$

Step 3. Reoptimization

$$\hat{p}_{i+1} = \mathcal{P}\left(\frac{\hat{\alpha}_{i+1}}{2\hat{\beta}_{i+1}}\right). \quad (6)$$

- **All models are wrong, but some are useful !**
  - Price consistency: converge to the optimal price
  - Regret growth rate optimality:
    - Theoretically: near optimal $O(\sqrt{T}(\log T)^2)$
    - Practically: robust numerical experiments performance

Table 1 The Impact of Misspecification

		Well specified			Misspecified					
Demand functions		$\mathcal{L}_1$ (linear)			$\mathcal{L}_2$ (exponential)			$\mathcal{L}_3$ (logit)		
		Time periods (T)			Time periods (T)			Time periods (T)		
	ρ	100	500	10^3	100	500	10^3	100	500	10^3
$\sigma = 0.25$	0.25	0.90	0.94	0.95	0.91	0.94	0.95	0.84	0.90	0.92
	0.5	0.87	0.93	0.95	0.93	0.96	0.96	0.87	0.93	0.95
	0.75	0.79	0.88	0.91	0.94	0.96	0.97	0.91	0.95	0.96
$\sigma = 0.5$										
	0.25	0.83	0.89	0.91	0.82	0.87	0.89	0.69	0.77	0.80
	0.5	0.80	0.88	0.91	0.87	0.92	0.93	0.76	0.84	0.87
	0.75	0.74	0.84	0.87	0.90	0.94	0.95	0.81	0.88	0.91

Notes. Fraction of optimal (oracle) revenues achieved by the linear-based pricing policy, averaged over a set of 500 random test instances. The standard error of the mean was always below 0.0125.

PART 2

Online KWSA algorithm

First, the classical KWSA algorithm is for solving offline optimization problems where only the terminal solutions of the iterations; Second, Algorithm 1 uses a forward finite-difference gradient estimator instead of a central finite-difference estimator.

2.1 Online stochastic optimization

- Formulation: $\min_{\mathbf{x}\in\Omega}\{ f(\mathbf{x}) = E(F(\mathbf{x},\xi))\}$
- Regret: $R(T) = \sum_{t=1}^{T} E(f(\mathbf{x_t}) - f(\mathbf{x}^*))$
- Assumption:
  - $\Omega \subset R^d$ compact & convex
  - $E(F(\mathbf{x},\xi)^2) \le M, \forall \mathbf{x} \in \Omega$
  - f twice differentiable and Hessian continuous
  - f strongly convex
  - $\mathbf{x}^* \in \mathrm{int}(\Omega)$

2.1 Online stochastic optimization

- A0: $\Omega \subset R^d$ compact & convex
  - compact (+A2) : the existence of $\mathbf{x}^*$
  - convex : the uniqueness of the projection $\Pi_\Omega(\mathbf{x})$
- A1: $E(F(\mathbf{x},\xi)^2) \le M, \forall \mathbf{x} \in \Omega$
  - convergence in mean square
- A2: f twice differentiable and Hessian continuous
  - Hessian continuous (+A0 compact): $\|\nabla^2 f(\mathbf{x})\| \le B_2, \forall \mathbf{x} \in \Omega$
- A3: f B_1-strongly convex
  - definition: $f(\mathbf{x_2}) \ge f(\mathbf{x_1}) + \nabla f(\mathbf{x_1})^T(\mathbf{x_2}-\mathbf{x_1}) + \frac{1}{2}B_1\|\mathbf{x_2}-\mathbf{x_1}\|^2, \forall \mathbf{x_1},\mathbf{x_2}$ (simplification: $\nabla^2 f(\mathbf{x}) - B_1\mathbf{I} \succeq \mathbf{0}$)
  - corollary 1: $[\nabla f(\mathbf{x}_2) - \nabla f(\mathbf{x_1})]^T(\mathbf{x_2}-\mathbf{x_1}) \ge B_1\|\mathbf{x_2}-\mathbf{x_1}\|^2, \forall \mathbf{x_1},\mathbf{x_2}$
  - corollary 2: the uniqueness of $\mathbf{x}^*$
- A4: $\mathbf{x}^* \in \mathrm{int}(\Omega)$
  - $\nabla f(\mathbf{x}^*) = 0$

2. 2 Online KWSA algorithm

ALGORITHM 1 Online KWSA algorithm

Initialization. Let $\mathbf{x}_0 \in \Omega$ be a starting solution. Let the iteration counter $n = 1$ and period counter $t = 0$.

Step 1. Function evaluations and information collection.

- Let $t = t + 1$. Set $\widetilde{\mathbf{x}}_t = \mathbf{x}_n$ and observe $F(\widetilde{\mathbf{x}}_t, \xi_{n,0})$;
- For $i = 1, 2, \ldots, d$,
  Let $t = t + 1$. Set $\widetilde{\mathbf{x}}_t = \mathbf{x}_n + c_n \mathbf{e}_i$ and observe $F(\widetilde{\mathbf{x}}_t, \xi_{n,i})$.
  End the for-loop.

Step 2. Updating.

Let

$$\mathbf{x}_{n+1} = \Pi_\Omega(\mathbf{x}_n - a_n \mathbf{G}(\mathbf{x}_n)),$$

where Π_Ω is a projection operator onto the set Ω, that is, $\Pi_\Omega(\mathbf{x}) = \operatorname{argmin}_{\mathbf{x}' \in \Omega} \|\mathbf{x} - \mathbf{x}'\|$, and

$$\mathbf{G}(\mathbf{x}_n) = \frac{1}{c_n}([F(\mathbf{x}_n + c_n \mathbf{e}_1, \xi_{n,1}) - F(\mathbf{x}_n, \xi_{n,0})], \ldots, [F(\mathbf{x}_n + c_n \mathbf{e}_d, \xi_{n,d}) - F(\mathbf{x}_n, \xi_{n,0})])^{\mathrm{T}}.$$

Let $n = n + 1$ and go back to Step 1.

- Step 1: calculate the (forward) finite-difference estimator (c_n sequence)
- Step 2: projection gradient descent (a_n sequence)

2. 2 Online KWSA algorithm

Theorem 1 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1–4 are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists a constant* $\lambda > 0$ *such that* $\mathrm{E}[\|\mathbf{x}_n - \mathbf{x}^*\|^2] \leq \lambda n^{-\frac{1}{2}}$ *for all* $n = 1, 2, \ldots$.

- a_n sequence: gradient descent step size
- f B_1-strongly convex (A3)

$$\nabla^2 f(\mathbf{x}) - B_1 \mathbf{I} \succeq \mathbf{0}$$

- c_n sequence: forward finite-difference

- Fabian (1967):
  - Forward finite-difference estimator:

$$a_n = \gamma n^{-1}, c_n = \delta n^{-\frac{1}{4}} \Rightarrow E(\|x_n - x^*\|^2) = O(n^{-\frac{1}{2}})$$

  - Central finite-difference estimator:

$$a_n = \gamma n^{-1}, c_n = \delta n^{-\frac{1}{6}} \Rightarrow E(\|x_n - x^*\|^2) = O(n^{-\frac{2}{3}})$$

(But in the online stochastic optimization, it doesn’t lower the growth rate of the expected cumulative regret !)

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} = E(\|\Pi_\Omega(\mathbf{x_n} - a_n G(\mathbf{x_n})) - \mathbf{x}^*\|^2)$$
$$= E(\|\Pi_\Omega(\mathbf{x_n} - a_n G(\mathbf{x_n})) - \Pi_\Omega(\mathbf{x}^*)\|^2)$$
$$\le E(\|\mathbf{x_n} - a_n G(\mathbf{x_n}) - \mathbf{x}^*\|^2)$$
$$= b_n + a_n^2 E(\|G(\mathbf{x_n})\|^2) - 2a_n E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*))$$

where $G(\mathbf{x_n}) = \frac{1}{c_n}([F(\mathbf{x_n} + c_n\mathbf{e_1}, \xi_{n,1}) - F(\mathbf{x_n}, \xi_{n,0}),\ ...,\ F(\mathbf{x_n} + c_n\mathbf{e}_d, \xi_{n,d}) - F(\mathbf{x_n}, \xi_{n,0})]^T$,

Define $g(\mathbf{x_n}) = E(G(\mathbf{x_n}) \mid \mathbf{x_n}) = \frac{1}{c_n}([f(\mathbf{x_n} + c_n\mathbf{e_1}) - f(\mathbf{x_n}),\ ...,\ f(\mathbf{x_n} + c_n\mathbf{e}_d) - f(\mathbf{x_n})]^T$.

Theorem 1 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1–4 are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists a constant* $\lambda > 0$ *such that* $\mathrm{E}[\|\mathbf{x}_n - \mathbf{x}^*\|^2] \le \lambda n^{-\frac{1}{2}}$ *for all* $n = 1, 2, \ldots$.

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- Define: $b_n = E(\|x_n - x^*\|^2)$
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$$b_{n+1} \le b_n + a_n^2 E(\|G(\mathbf{x_n})\|^2) - 2a_n E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*))$$

$$\begin{aligned} E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)) &= E[E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)) \mid \mathbf{x_n}] \\ &= E[g(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)] \\ &= E[\nabla f(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)] + E[(g(\mathbf{x_n}) - \nabla f(\mathbf{x_n}))^T(\mathbf{x_n} - \mathbf{x}^*)] \end{aligned}$$

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2. 2 Online KWSA algorithm

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A4 corollary: $\nabla f(\mathbf{x}^*) = 0$

A3 corollary 1: $[\nabla f(\mathbf{x}_2) - \nabla f(\mathbf{x_1})]^T(\mathbf{x_2} - \mathbf{x_1}) \ge B_1 \|\mathbf{x_2} - \mathbf{x_1}\|^2, \forall \mathbf{x_1}, \mathbf{x_2}$

$$\begin{aligned} E[\nabla f(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)] &= E[(\nabla f(\mathbf{x_n}) - \nabla f(\mathbf{x}^*))^T(\mathbf{x_n} - \mathbf{x}^*)] \\ &\ge B_1 E[\|\mathbf{x_n} - \mathbf{x}^*\|^2] = B_1 b_n \end{aligned}$$

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} \le b_n + a_n^2 E(\|G(\mathbf{x_n})\|^2) - 2a_n E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*))$$

$$E(G(\mathbf{x_n})^T(\mathbf{x_n} - \mathbf{x}^*)) \ge B_1 b_n + E[(g(\mathbf{x_n}) - \nabla f(\mathbf{x_n}))^T(\mathbf{x_n} - \mathbf{x}^*)]$$

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A2 corollary: $\|\nabla^2 f(\mathbf{x})\| \le B_2, \forall \mathbf{x} \in \Omega$

$$g(\mathbf{x_n}) = E(G(\mathbf{x_n}) \mid \mathbf{x_n}) = \frac{1}{c_n}([f(\mathbf{x_n} + c_n\mathbf{e_1}) - f(\mathbf{x_n}),\ \ldots,\ f(\mathbf{x_n} + c_n\mathbf{e}_d) - f(\mathbf{x_n})]^T$$

$$g(\mathbf{x_n}) - \nabla f(\mathbf{x_n}) = \frac{1}{2}c_n(\mathbf{e_1^T}\nabla^2 f(\boldsymbol{\eta}_1)\mathbf{e}_1,\ \ldots,\ \mathbf{e_d^T}\nabla^2 f(\boldsymbol{\eta}_\mathbf{d})\mathbf{e_d})^T \le \frac{1}{2}B_2 c_n \mathbf{1}$$

$$E[(g(\mathbf{x_n}) - \nabla f(\mathbf{x_n}))^T(\mathbf{x_n} - \mathbf{x}^*)] \ge -\frac{1}{2}B_2 c_n E[\|\mathbf{x_n} - \mathbf{x}^*\|_1] \ge -\frac{1}{2}B_2 c_n E[\sqrt{d}\,\|\mathbf{x_n} - \mathbf{x}^*\|_2]$$

$$\ge -\frac{1}{2}B_2 c_n \sqrt{d}\sqrt{b_n}$$

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
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$$E(G(\mathbf{x_n})^T (\mathbf{x_n} - \mathbf{x}^*)) \ge B_1 b_n - \frac{1}{2} B_2 c_n \sqrt{d} \sqrt{b_n}$$

Theorem 1 *Suppose that Algorithm* 1 *is used to solve Problem* (1) *and Assumptions* 1–4 *are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists a constant* $\lambda > 0$ *such that* $\mathrm{E}[\|\mathbf{x}_n - \mathbf{x}^*\|^2] \le \lambda n^{-\frac{1}{2}}$ *for all* $n = 1, 2, \ldots$.

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A1: $E(F(\mathbf{x},\xi)^2) \le M, \forall \mathbf{x} \in \Omega$

$$G(\mathbf{x_n}) = \frac{1}{c_n}([F(\mathbf{x_n} + c_n\mathbf{e_1}, \xi_{n,1}) - F(\mathbf{x_n}, \xi_{n,0}),\ \ldots,\ F(\mathbf{x_n} + c_n\mathbf{e}_d, \xi_{n,d}) - F(\mathbf{x_n}, \xi_{n,0})]^T$$

$$E(\|G(\mathbf{x_n})\|^2) \le \frac{4dM^2}{c_n^2} \quad (\ \|\mathbf{a}+\mathbf{b}\| \le \max\{2\|\mathbf{a}\|, 2\|\mathbf{b}\|\}\ )$$

$$b_{n+1} \le (1 - 2a_n B_1) b_n + a_n c_n B_2 \sqrt{d}\sqrt{b_n} + \frac{4da_n^2 M^2}{c_n^2}$$

$$= (1 - \frac{2\gamma B_1}{n}) b_n + \gamma\delta B_2 \sqrt{d}\, n^{-\frac{5}{4}} \sqrt{b_n} + \frac{4d\gamma^2 M^2}{\delta^2} n^{-\frac{3}{2}}$$

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} \le (1 - \frac{2\gamma B_1}{n})b_n + \gamma\delta B_2\sqrt{d}\,n^{-\frac{5}{4}}\sqrt{b_n} + \frac{4d\gamma^2 M^2}{\delta^2} n^{-\frac{3}{2}}$$

- Induction (Appendix):

$$b_n \le \lambda n^{-\frac{1}{2}},\ n = 1, 2, 3...$$

Theorem 1 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1–4 are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$*. Then, there exists a constant* $\lambda > 0$ *such that* $E[\|\mathbf{x}_n - \mathbf{x}^*\|^2] \le \lambda n^{-\frac{1}{2}}$ *for all* $n = 1, 2, \ldots$.

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} \le (1 - \frac{2\gamma B_1}{n})b_n + \gamma\delta B_2\sqrt{d}\,n^{-\frac{5}{4}}\sqrt{b_n} + \frac{4d\gamma^2 M^2}{\delta^2} n^{-\frac{3}{2}}$$

- Proof (Appendix):

Let $\alpha = 2\gamma B_1$, $\beta = \gamma\delta B_2\sqrt{d}$, and $\omega = \frac{4d\gamma^2M^2}{\delta^2}$. By Equation (11), we have

$$b_{n+1} \le \left(1 - \frac{\alpha}{n}\right) b_n + \beta n^{-\frac{5}{4}}\sqrt{b_n} + \omega n^{-\frac{3}{2}}. \qquad \text{(A1)}$$

Let $\lambda = \max\{b_1, \lambda_0\}$, where

$$\lambda_0 = \left(\frac{\beta + \sqrt{\beta^2 + 2\omega(2\alpha - 1)}}{2\alpha - 1}\right)^2$$

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} \le (1 - \frac{2\gamma B_1}{n}) b_n + \gamma \delta B_2 \sqrt{d} n^{-\frac{5}{4}} \sqrt{b_n} + \frac{4d\gamma^2 M^2}{\delta^2} n^{-\frac{3}{2}}$$

- Proof (Appendix):

and, because $2\alpha - 1 > 0$, λ_0 also satisfies

$$(2\alpha - 1)\kappa - 2\beta\sqrt{\kappa} - 2\omega \ge 0, \qquad \forall \kappa \ge \lambda_0. \qquad \text{(A2)}$$

We prove by induction that $b_n \le \lambda n^{-\frac{1}{2}}$. It is easy to see that it holds for $n = 1$. For any $n = 1, 2, \ldots$, suppose that $b_n \le \lambda n^{-\frac{1}{2}}$. Then, by Equation (A1) and because $1 - \alpha/n > 0$ due to $\alpha < 1$,

$$\begin{aligned} b_{n+1} &\le \left(1 - \frac{\alpha}{n}\right) \lambda n^{-\frac{1}{2}} + \beta\sqrt{\lambda} n^{-3/2} + \omega n^{-\frac{3}{2}} \\ &= \lambda n^{-\frac{1}{2}} - (\alpha\lambda - \beta\sqrt{\lambda} - \omega) n^{-\frac{3}{2}} \\ &= \lambda n^{-\frac{1}{2}} - \frac{\lambda}{2} n^{-\frac{3}{2}} - \frac{1}{2}[(2\alpha - 1)\lambda - 2\beta\sqrt{\lambda} - 2\omega] n^{-\frac{3}{2}} \\ &\le \lambda \left(n^{-\frac{1}{2}} - \frac{1}{2} n^{-\frac{3}{2}}\right), \qquad \text{(A3)} \end{aligned}$$

2. 2 Online KWSA algorithm

- Define: $b_n = E(\|x_n - x^*\|^2)$
- Recurrence relation:

$$b_{n+1} \le (1 - \frac{2\gamma B_1}{n}) b_n + \gamma \delta B_2 \sqrt{d} n^{-\frac{5}{4}} \sqrt{b_n} + \frac{4 d \gamma^2 M^2}{\delta^2} n^{-\frac{3}{2}}$$

- Proof (Appendix):

where the last inequality follows from Equation (A2) and the fact that $\lambda = \max\{b_0, \lambda_0\} \ge \lambda_0$. Let $g(x) = x^{-\frac{1}{2}}$. Then, $g'(x) = -\frac{1}{2} x^{-\frac{3}{2}}$. Notice that $g(x)$ is convex. Then,

$$g(x') - g(x) \ge g'(x)(x' - x).$$

Then,

$$(n+1)^{-\frac{1}{2}} - n^{-\frac{1}{2}} = g(n+1) - g(n) \ge g'(n) = -\frac{1}{2} n^{-\frac{3}{2}}.$$

Therefore,

$$n^{-\frac{1}{2}} - \frac{1}{2} n^{-\frac{3}{2}} \le (n+1)^{-\frac{1}{2}}.$$

Then, by Equation (A3), we have $b_{n+1} \le \lambda (n+1)^{-\frac{1}{2}}$. This concludes the induction proof and, therefore, $b_n \le \lambda n^{-\frac{1}{2}}$ for all $n = 1, 2, \ldots$.

2. 2 Online KWSA algorithm

Theorem 2 *Suppose that Algorithm* 1 *is used to solve Problem* (1) *and Assumptions* 1-4 *are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists constant* $\kappa_1 > 0$ *and* $\kappa_2 > 0$ *such that* $R(T) \leq \kappa_1 \sqrt{T} + \kappa_2$ *for all* $T = 1, 2, \ldots$.

- Step 1: calculate the (forward) finite-difference estimator (c_n sequence)
- Step 2: projection gradient descent (a_n sequence)

- Regret:

$$r(n) = E[F(\mathbf{x_n}, \xi_{n,0}) + F(\mathbf{x_n} + c_n\mathbf{e_1}, \xi_{n,1}) + \ldots\ + F(\mathbf{x_n} + c_n\mathbf{e_d}, \xi_{n,d})] - (d+1) f(\mathbf{x}^*)$$

$$= E[f(\mathbf{x_n}) - f(\mathbf{x}^*)] + E[f(\mathbf{x_n} + c_n\mathbf{e_1}) - f(\mathbf{x}^*)] + \ldots\ + E[f(\mathbf{x_n} + c_n\mathbf{e_d}) - f(\mathbf{x}^*)]$$

2. 2 Online KWSA algorithm

- Regret:

$$r(n)=E[f(\mathbf{x_n})-f(\mathbf{x}^*)]+E[f(\mathbf{x_n}+c_n\mathbf{e_1})-f(\mathbf{x}^*)]+\dots$$
$$+E[f(\mathbf{x_n}+c_n\mathbf{e_d})-f(\mathbf{x}^*)]$$

Theorem 2 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1-4 are satisfied. Let* $a_n=\gamma n^{-1}$ *and* $c_n=\delta n^{-\frac{1}{4}}$ *with* $1/(4B_1)<\gamma<1/(2B_1)$ *and* $\delta>0$. *Then, there exists constant* $\kappa_1>0$ *and* $\kappa_2>0$ *such that* $R(T)\le\kappa_1\sqrt{T}+\kappa_2$ *for all* $T=1,2,\dots$.

A2 corollary: $\left\|\nabla^2 f(\mathbf{x})\right\|\le B_2,\forall\mathbf{x}\in\Omega$

A4 corollary: $\nabla f(\mathbf{x}^*)=0$

$$f(\mathbf{x_n})-f(\mathbf{x}^*)\le\frac{1}{2}B_2\left\|\mathbf{x_n}-\mathbf{x}^*\right\|^2$$

$$f(\mathbf{x_n}+c_n\mathbf{e_i})-f(\mathbf{x}^*)\le\frac{1}{2}B_2\left\|\mathbf{x_n}+c_n\mathbf{e_i}-\mathbf{x}^*\right\|^2\le B_2(\left\|\mathbf{x_n}-\mathbf{x}^*\right\|^2+c_n^2)$$

Theorem 1: $b_n\le\lambda n^{-\frac{1}{2}},\ n=1,2,3\dots$

$$r(n)\le(\frac{1}{2}+d)B_2b_n+dB_2c_n^2$$
$$\le(\frac{2d+1}{2}\lambda+d\delta^2)B_2n^{-\frac{1}{2}}$$

2. 2 Online KWSA algorithm

- Regret:

$$r(n) \le (\frac{1}{2} + d)B_2 b_n + dB_2 c_n^2$$

$$\le (\frac{2d+1}{2}\lambda + d\delta^2)B_2 n^{-\frac{1}{2}}$$

- If central finite-difference estimator:

$$c_n = \delta n^{-\frac{1}{6}} \Rightarrow r(n) \le O(n^{-\frac{1}{3}})$$

not as good as $O(n^{-\frac{1}{2}})$

Theorem 2 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1-4 are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists constant* $\kappa_1 > 0$ *and* $\kappa_2 > 0$ *such that* $R(T) \le \kappa_1 \sqrt{T} + \kappa_2$ *for all* $T = 1, 2, \ldots$.

2. 2 Online KWSA algorithm

- Regret:

$$r(n) \le (\frac{2d+1}{2}\lambda + d\delta^2)B_2 n^{-\frac{1}{2}}$$

$$R(T) \le \sum_{n=1}^{\lceil T/(d+1)\rceil} r(n) = \left(\frac{2d+1}{2}\lambda + d\delta^2\right) B_2 \sum_{n=1}^{\lceil T/(d+1)\rceil} n^{-\frac{1}{2}}$$
$$\le \left(\frac{2d+1}{2}\lambda + d\delta^2\right) B_2 \int_0^{(T+d)/(d+1)} x^{-\frac{1}{2}} dx$$
$$\le \frac{[(2d+1)\lambda + 2d\delta^2]B_2}{\sqrt{d+1}}(\sqrt{T} + \sqrt{d})$$
$$\le 2(\lambda + \delta^2)\sqrt{d+1} \cdot \sqrt{T} + 2(\lambda + \delta^2)(d+1).$$

Theorem 2 *Suppose that Algorithm 1 is used to solve Problem (1) and Assumptions 1-4 are satisfied. Let* $a_n = \gamma n^{-1}$ *and* $c_n = \delta n^{-\frac{1}{4}}$ *with* $1/(4B_1) < \gamma < 1/(2B_1)$ *and* $\delta > 0$. *Then, there exists constant* $\kappa_1 > 0$ *and* $\kappa_2 > 0$ *such that* $R(T) \le \kappa_1 \sqrt{T} + \kappa_2$ *for all* $T = 1, 2, \ldots$.

PART 3

Multi-product dynamic pricing and numerical experiments

Theorem 3 essentially shows that the KW pricing policy is asymptotically optimal. The nonparametric approach may avoid model misspecifications that always exist in parametric models.

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3. 1 KW pricing policy

- Demand: $\mathbf{D}(\mathbf{p}) = (D_1(\mathbf{p}), D_2(\mathbf{p}), \ldots, D_d(\mathbf{p}))^T$
- Maximizing the revenue:

$$\mathbf{p}^* = \underset{\mathbf{p}\in\Omega}{\operatorname{argmax}} \{\theta(\mathbf{p}) = E[\mathbf{p}^{\mathbf{T}}\mathbf{D}(\mathbf{p})]\}$$

- Regret:

$$R(\pi, T) = \sum_{t=1}^{T} E(\theta(\mathbf{p}^*) - \theta(\mathbf{p}_\mathbf{t}))$$

3. 1 KW pricing policy

ALGORITHM 2 KW pricing policy

Initialization. Let $\mathbf{p}_0 \in \Omega$ by a starting price vector. Let the iteration counter $n = 1$ and period counter $t = 0$.

Step 1. Pricing and information collection.

- Let $t = t + 1$. Set $\widetilde{\mathbf{p}}_t = \mathbf{p}_n$ and observe $\Theta_t = \Theta_t(\widetilde{\mathbf{p}}_t)$;
- For $i = 1, 2, \ldots, d$,

Let $t = t + 1$. Set $\widetilde{\mathbf{p}}_t = \mathbf{p}_n + c_n \mathbf{e}_i$ and observe $\Theta_t = \Theta_t(\widetilde{\mathbf{p}}_t)$.

End the for-loop.

Step 2. Updating.

Let

$$\mathbf{p}_{n+1} = \Pi_\Omega(\mathbf{p}_n + a_n \mathbf{G}(\mathbf{p}_n)),$$

where Π_Ω is a projection operator onto the set Ω, that is, $\Pi_\Omega(\mathbf{p}) = \text{argmin}_{\mathbf{p}' \in \Omega} \|\mathbf{p} - \mathbf{p}'\|$, and

$$\mathbf{G}(\mathbf{p}_n) = \frac{1}{c_n}[(\Theta_{k+2} - \Theta_{k+1}), \ldots, (\Theta_{k+d+1} - \Theta_{k+1})]^{\mathrm{T}}$$

with $k = n(d+1)$. (Note that the index k introduced here is only for notational simplification of the expressions of Θ_t, where $t = k+1, \ldots, k+d+1$. In order to make the expression of $\mathbf{G}(\mathbf{p}_n)$ in Algorithm 2 consistent with that of $\mathbf{G}(\mathbf{x}_n)$ Algorithm 1, in fact, we can rewrite $\mathbf{G}(\mathbf{x}_n) = \frac{1}{c_n}([F(\widetilde{\mathbf{x}}_{k+2}, \xi_{n,1}) - F(\widetilde{\mathbf{x}}_{k+1}, \xi_{n,0})], \ldots, [F(\widetilde{\mathbf{x}}_{k+d+1}, \xi_{n,d})$ $([-F(\widetilde{\mathbf{x}}_{k+1}, \xi_{n,0})])^{\mathrm{T}}$, with $k = n(d+1)$.) Let $n = n + 1$ and go back to Step 1.

- Step 1: calculate the (forward) finite-difference
- Step 2: projection gradient descent

3. 1 KW pricing policy

Theorem 3 *Suppose that the KW pricing policy is used to solve Problem* (15) *and that the following assumptions hold*:

1. $\Omega \subset \Re^d$ *is a convex and compact set and* $\mathbf{p}^* \in \mathrm{int}(\Omega)$;
2. $\mathrm{E}[\mathbf{D}(\mathbf{p})]$ *is twice continuously differentiable in* Ω *and* $\max_{\mathbf{p}\in\Omega}\mathrm{E}[\|\mathbf{D}(\mathbf{p})\|^2] < \infty$;
3. $\theta(\mathbf{p})$ is strongly concave.

Then, there exist constants $\lambda > 0$, $\kappa_1 > 0$, *and* $\kappa_2 > 0$ *such that* $\mathrm{E}(\|\mathbf{p}_n - \mathbf{p}^*\|^2) \le \lambda n^{-\frac{1}{2}}$ *for all* $n = 1, 2, \ldots$ *and* $R(T, \Psi^{\mathrm{KW}}) \le \kappa_1\sqrt{T} + \kappa_2$ *for all* $T = 1, 2, \ldots$.

- Assumption:
  - $\Omega \subset R^d$ compact & convex
  - $E(F(\mathbf{x}, \xi)^2) \le M, \forall \mathbf{x} \in \Omega$
  - f twice differentiable and Hessian continuous
  - f strongly convex
  - $\mathbf{x}^* \in \mathrm{int}(\Omega)$

3. 2 Numerical experiments

- Experiment 1: illustration of the rate optimality of the total regret

$$\mathbf{D}_t(\mathbf{p}_t) = \alpha + \mathbf{B}\mathbf{p}_t + \epsilon_t, \quad \text{for } t = 1, 2, \ldots ,$$

where $\epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{dt})^{\mathrm{T}} \in \Re^n$, $\alpha = (\alpha_1, \ldots, \alpha_d)^{\mathrm{T}} \in \Re^d$, and $\mathbf{B}$ is a $d \times d$ matrix

$$\mathbf{B} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1d} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{d1} & \beta_{d2} & \cdots & \beta_{dd} \end{bmatrix}.$$

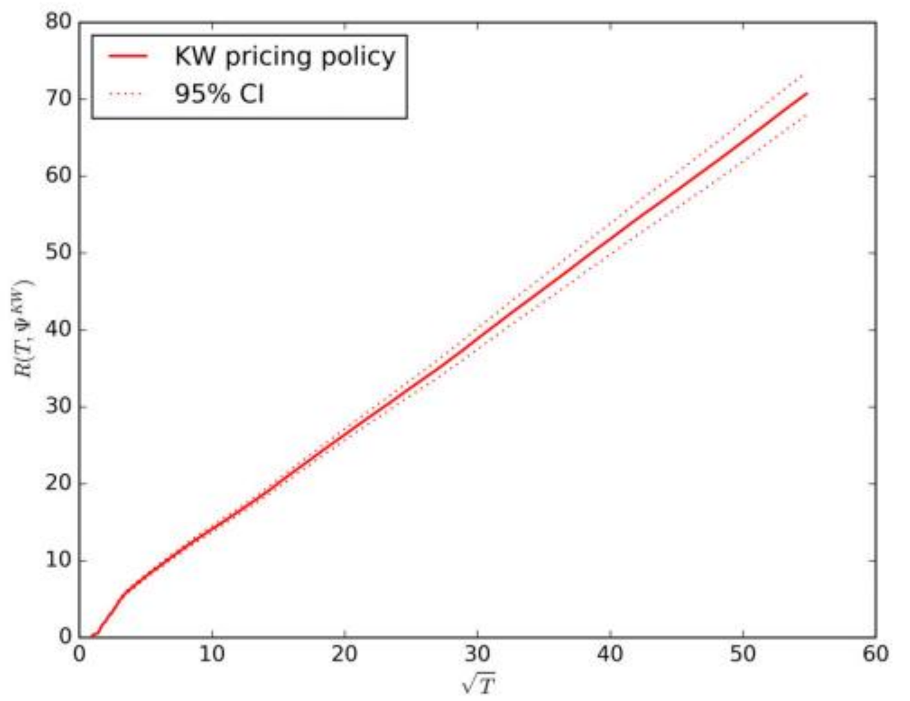

FIGURE 1 Average cumulative regret of the KW pricing policy [Colour figure can be viewed at wileyonlinelibrary.com]

3. 2 Numerical experiments

- Experiment 2: comparison with multi-product parametric pricing policies
  - Multi-product linear demand model
  - compared with ILS & MCILS (Keskin and Zeevi (2014))

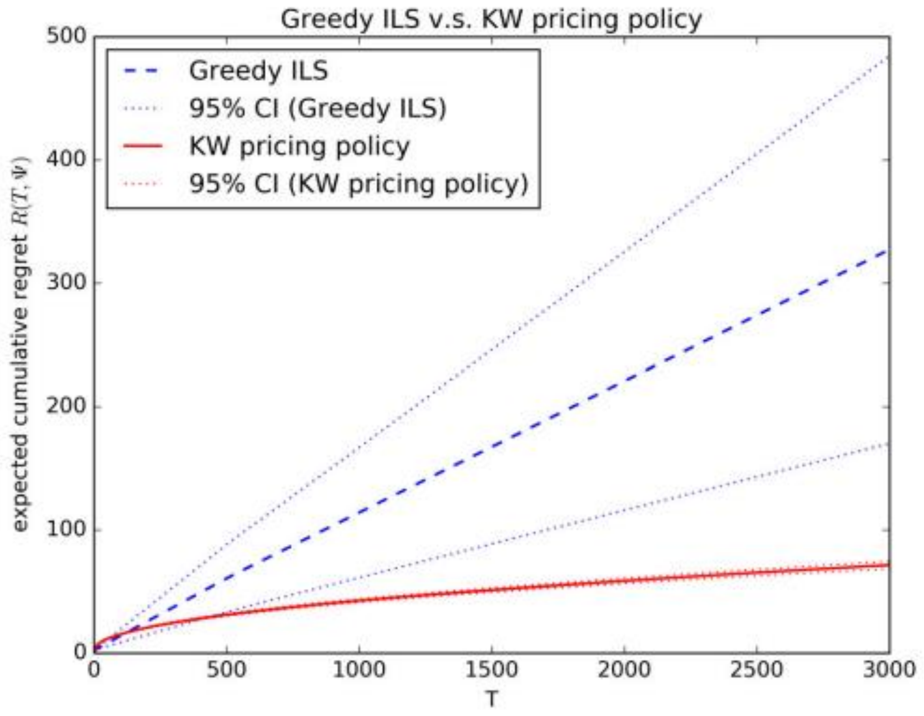

FIGURE 2 Comparisons between the ILS and KW pricing policies [Colour figure can be viewed at wileyonlinelibrary.com]

3. 2 Numerical experiments

- Experiment 3: comparison with a single-product nonparametric pricing policy

a. *Linear*: $\lambda(p) = (\alpha - \beta p)^+$, where $\alpha = 1$, $\beta = 0.5$, and $\sigma^2 = 0.05^2$. Let $\Omega = [l, u] = [0.5, 1.5]$, and then the optimal price is $p^* = 1$. Let $\widehat{p}_1 = 0.7$, $I_i = 1$, $\delta_t = \rho t^{-1/4}$ where $\rho = 0.5$, and $\mathcal{T}_i = \{1, \dots, t_{i+1}\}$ for the semimyopic policy; and let $p_0 = 0.7$, $\gamma = 3$, and $\delta = 1$ for the KW pricing policy.

b. *Exponential*: $\lambda(p) = \exp(\alpha - \beta p)$, where $\alpha = 1$, $\beta = 0.3$, and $\sigma^2 = 0.05^2$. Let $\Omega = [l, u] = [2.5, 3.5]$, and then the optimal price is $p^* = 3$. Let $\widehat{p}_1 = 2.7$, $I_i = 1$, $\delta_t = \rho t^{-1/4}$ where $\rho = 0.2$, and $\mathcal{T}_i = \{1, \dots, t_{i+1}\}$ for the semimyopic policy; and let $p_0 = 2.7$, $\gamma = 3$, and $\delta = 1$ for the KW pricing policy.

c. *Logit*: $\lambda(p) = \exp(\alpha - \beta p)/(1 + \exp(\alpha - \beta p))$, where $\alpha = 1$, $\beta = 0.3$, and $\sigma^2 = 0.05^2$. Let $\Omega = [l, u] = [3, 7]$, and then the ptimal price is $p^* \approx 5.2238$. Let $\widehat{p}_1 = 4.5$, $I_i = 1$, $\delta_t = \rho t^{-1/4}$ where $\rho = 0.5$, and $\mathcal{T}_i = \{1, \dots, t_{i+1}\}$ for the semimyopic policy; and let $p_0 = 4.5$, $\gamma = 10$, and $\delta = 1$ for the KW pricing policy.

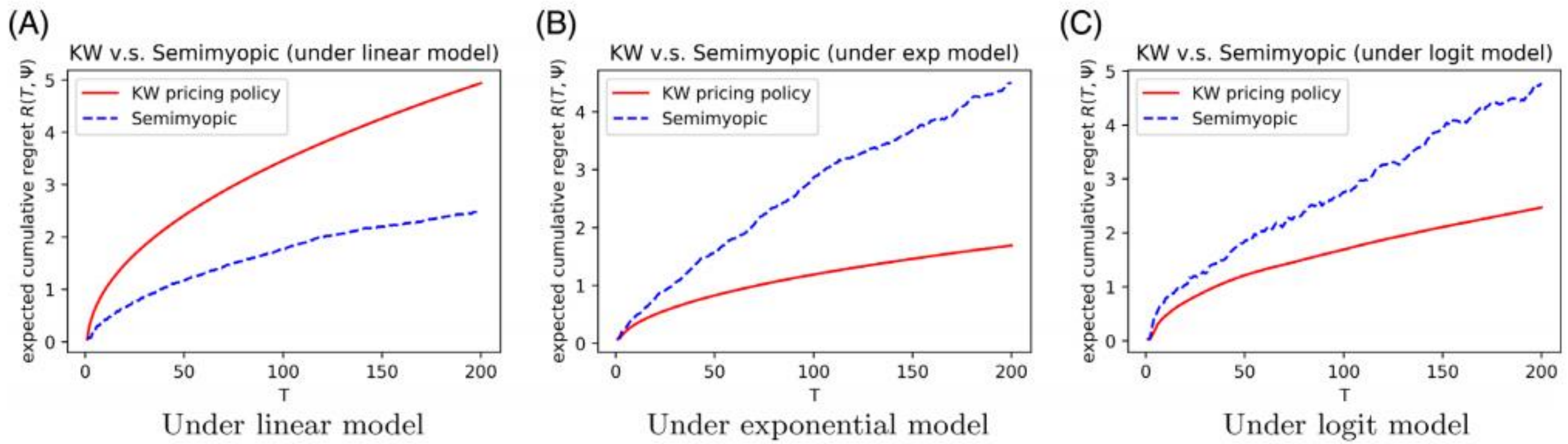

FIGURE 4 Comparisons between the semimyopic and KW pricing policies. (A) Under linear model. (B) Under exponential model [Colour figure can be viewed at wileyonlinelibrary.com]

3. 2 Numerical experiments

- Experiment 3: comparison with a single-product nonparametric pricing policy

- Besbes and Zeevi (2015)
  - Price consistency: converge to the optimal price
  - Regret growth rate optimality:
    - Theoretically: near optimal
    - Practically: robust numerical experiments performance

- Hong LJ, Li C & Luo J (2021)
  - Price consistency: converge to the optimal price
  - Regret growth rate optimality:
    - Theoretically: optimal
    - Practically: more robust
  - Applicability: much more general

THANKS!