Stochastic Trust-Region Response-Surface Method (STRONG) – A New Response-Surface Framework for Simulation Optimization

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Problem Definition

Consider the following simulation optimization problem:

$$\min_{x \in \mathbb{X}^p} \mathbb{E}[G(x)],$$

where

- x is the vector of continuous decision variables;
- $G(x)$ is the stochastic response evaluated at x;
- $\mathbb{X}^p$ is the p-dimensional decision space (or "region of operability" in terminology of the RSM)

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Trust region

- A trust region at a solution x' with a radius $\Delta > 0$ is

$$B(x', \Delta) = \{x \in \mathbb{X}^p : \|x - x'\| \leq \Delta\}$$

Assumptions

1. The objective function $g(x)$ is bounded below, twice differentiable, and there exist two positive constants, α_1 and β_1, such that $\|\nabla g(x)\| \leq \alpha_1$ and $\|H(x)\| \leq \beta_1$.
2. The estimators of $g(x)$ and $\nabla g(x)$ satisfy $\sup_{x \in \mathbb{X}^p} |\bar{G}(x,n) - g(x)| \to 0 \ w.p.1$ as $n \to \infty$, and $\sup_{x \in \mathbb{X}^p} |\bar{D}(x,m) - \nabla g(x)| \to 0 \ w.p.1$ as $m \to \infty$.

First- and second-order models

First- and second-order model at iteration k:

$$r_k(x) = \hat{g}_k(x_k) + \hat{\nabla} g_k^T(x_k)(x - x_k),$$

$$\mathsf{r}_k(x) = \hat{g}_k(x_k) + \hat{\nabla} g_k^T(x_k)(x - x_k) + \frac{1}{2}(x - x_k)^T \hat{H}_k(x_k)(x - x_k).$$

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Cauchy Point Calculation:

1. Find the steepest descent direction $d_k = argmin\{\hat{g}_k(x_k) + \hat{\nabla} g_k^T(x_k) d : \|d\| \leq \Delta_k\}$;
2. Choose a step size $\tau_k = argmin\{r_k(\tau d_k) : \tau > 0, \|\tau d_k\| \leq \Delta_k\}$.
3. Let $x_k^* = x_k + \tau_k d_k$.

RC (Ratio-Comparison) test

The RC test computes

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Let $0 < \eta_0 < \eta_1 < 1$, for example, set $\eta_0 = 1/4, \eta_1 = 3/4$.

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- If ρ_k is close to 0 or negative ($\rho_k < \eta_0$), which implies that the "observed reduction" does not agree with the "predicted reduction", the new solution will be rejected.

SR (Sufficient-Reduction) test

The SR test is defined as

$$H_0 : g(x_k) - g(x_k^*) \leq \eta_0^2 \zeta_k \quad v.s. \quad H_1 : g(x_k) - g(x_k^*) > \eta_0^2 \zeta_k,$$

Let $0 < \eta_0 < \eta_1 < 1$, for example, set $\eta_0 = 1/4, \eta_1 = 3/4.$ where

$$\begin{aligned} \zeta_k &:= \|\hat{\nabla} g_k(x_k)\| \Delta_k, \\ \zeta_k &:= \frac{1}{2} \|\hat{\nabla} g_k(x_k)\| \min\{\frac{\|\hat{\nabla} g_k(x_k)\|}{\hat{\|} H_k(x_k)\|}, \Delta_k\}. \end{aligned}$$

in first- or second-order models respectively. If H_0 is rejected, we then conclude that the new solution yields a sufficient reduction.

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3. Simulate several observations at x_k^* and estimate $g(x_k^*)$;
4. Conduct ratio-comparison (RC) and sufficient-reduction (SR) tests to examine the quality of x^k and to update x_{k+1} and the size of trust region Δ_{k+1}.

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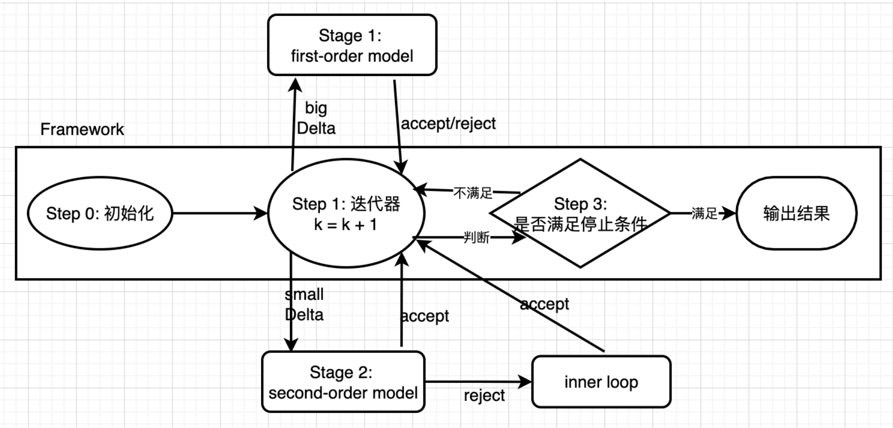

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- Step 0. Set the iteration count $k = 0$. Select an initial solution x_0, initial sample size of the center point n_0 and of the gradient estimator m_0, initial trust region size Δ_0, the switch threshold $\tilde{\Delta}$ satisfying $\Delta_0 > \tilde{\Delta} > 0$, and constants η_0, η_1, γ_1, and γ_2 satisfying $0 < \eta_0 < \eta_1 < 1$ and $0 < \gamma_1 < 1 < \gamma_2$.

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- Step 2. If the termination criterion is satisfied, stop and return the solution. Otherwise, go to Step 1.

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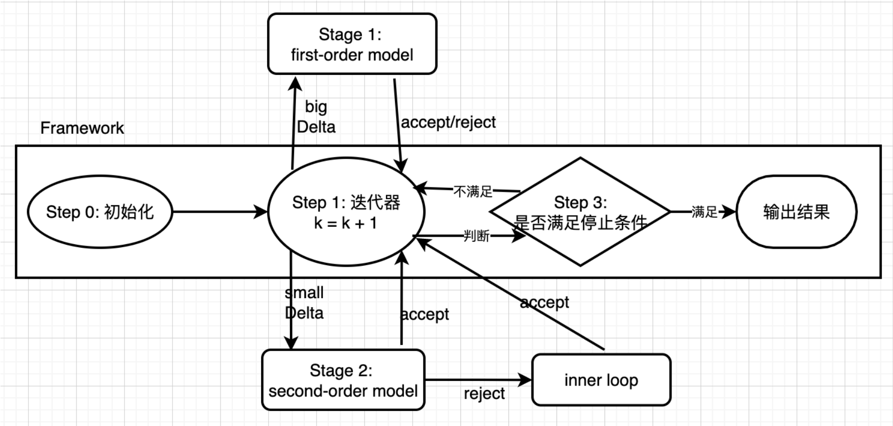

Thanks

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